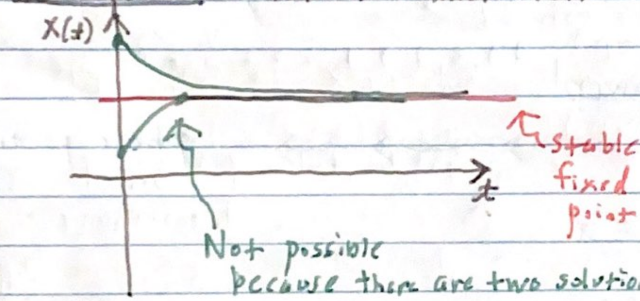
Lecture 3: Existence and Uniqueness

Theorem - Consider the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0.$$

Suppose $f'(x)$ is continuous on an open interval R containing x_0. Then the initial value problem has a solution $x(t)$ on some interval $(-\tau, \tau)$ about $t=0$, and the solution is unique.

Consequence - Solutions cannot cross

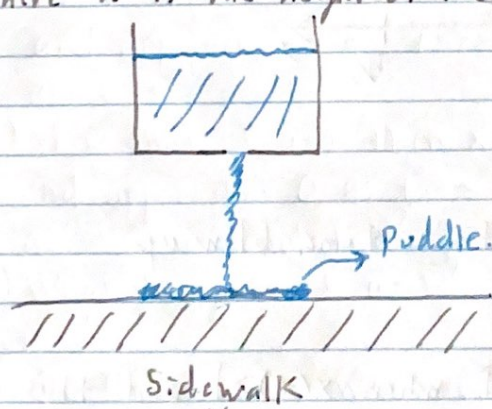


Example:

A leaky bucket can be modeled by

$$\dot{h} = -Ch^{1/2}, \rightarrow \text{comes from conservation of energy } \tfrac{1}{2}v^2 = -mgh.$$

where h is the height of the water in the bucket



$$\dot{h} = -Ch^{1/2}$$
$$h(0) = 0$$

Infinite number of solutions

$$h(t) = \begin{cases} \left(\dfrac{C(\tau - t)}{2}\right)^2 & t \le \tau \\ 0 & t > \tau \end{cases},$$

where $\tau \le 0$. Lets check:

$$h'(t) = \begin{cases} \dfrac{2c(\tau - t)}{2} \cdot \dfrac{-c}{2}, & t \le \tau \\ 0, & t > \tau \end{cases}$$

$$= \begin{cases} -c \cdot c\left(\dfrac{\tau - t}{2}\right), & t \le \tau \\ 0, & t \ge \tau \end{cases}$$

$$= -c h^{1/2}.$$

Solutions are not unique. The time τ when the bucket emptied is impossible to determine from looking at the puddle.

<u>Finite time blowup</u>

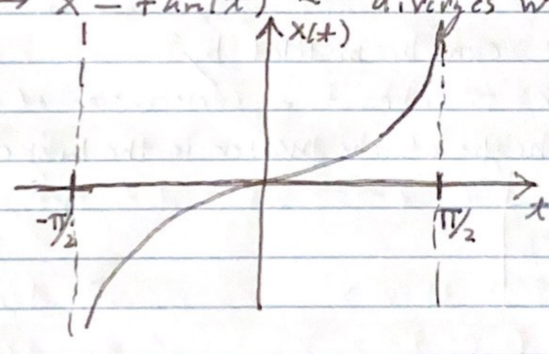
$\dot{x} = 1 + x^2$

$x(0) = 0$

$\Rightarrow \int_0^x \frac{1}{1+x^2}\,dx = t$

$\Rightarrow \tan^{-1}(x) = t$

$\Rightarrow x = \tan(t)$ ← diverges when $t = \pi/2$



$\dot{x} = 1 + x^4$ also has finite time blow up

$x(0) = 0$

There exists time T when $x(t) \ge 1$ and thus for $t \ge T$:

$$\dot{x} = 1 + x^4 \ge 1 + x^2$$

and thus $x \to \infty$.