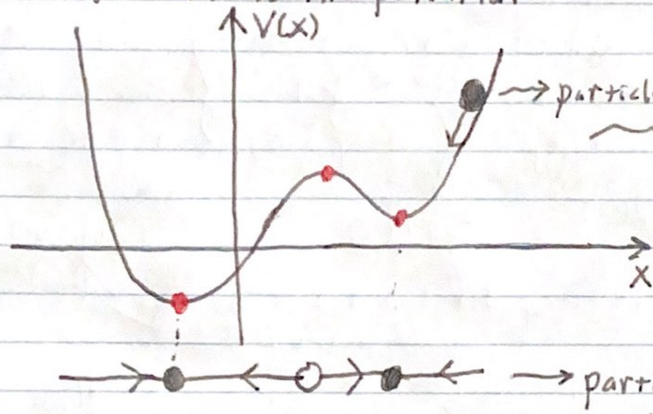
Lecture #4: Potentials and Impossibility of Oscillations

Potential Systems (Gradient System)

$$\dot{x} = -\frac{dV}{dx}$$

$V: \mathbb{R} \to \mathbb{R}$ is the potential



→ particle rolling down hill.

→ fixed points where $\frac{dV}{dx} = 0$

→ particle rolls down to minimum.

Theorem - $\frac{dV}{dt} \leq 0$ and $\frac{dV}{dt} = 0$ when $\frac{dV}{dx} = 0$.

~~proof~~:

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} \quad \text{(Chain Rule)}$$

$$= \frac{dV}{dx}\left(-\frac{dV}{dx}\right)$$

$$= -\left(\frac{dV}{dx}\right)^2.$$

■

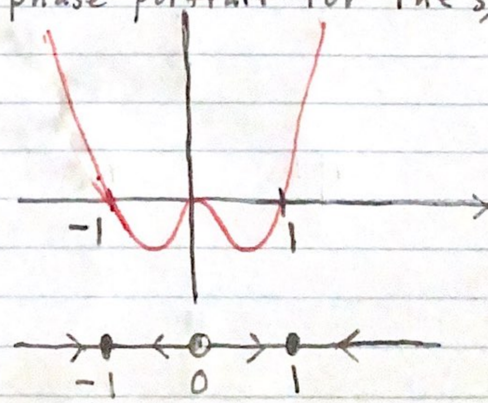
Example:

Sketch the potential and phase portrait for the system

$$\dot{x} = x - x^3$$

$$\Rightarrow -\frac{dV}{dx} = x - x^3$$

$$\Rightarrow V(x) = \frac{x^4}{4} - \frac{x^2}{2}$$



Theorem - All 1-d autonomous system of the form $\dot{x} = f(x)$ are gradients and thus cannot have oscillations.

~~proof~~:

If $\dot{x} = f(x)$ then $V(x) = -\int_0^x f(s)\,ds$ is a potential.