Lecture 5: SIS Model and Transcritical Bifurcations

SIS Model:

$S \sim$ # of infected individuals

$I \sim$ # of infected individuals

Assumptions:

$$\dot{S} = -f(S,I) + g(I), \quad f = \text{rate of infection}$$
$$\dot{I} = f(S,I) - g(I) \quad g = \text{rate of recovery}$$

$$f(0,I) = 0$$
$$f(S,0) = 0$$
$$g(0) = 0$$

The simplest functions that work are of the form:

- $f(S,I) = \alpha SI = (\alpha I)S$

  $[\alpha] = \frac{1}{\text{pop}\cdot\text{time}}$, $[\alpha I] = \frac{1}{\text{time}}$ = infection rate that depends on # of infected.

- $g(I) = \beta I$

  $[\beta] = \frac{1}{\text{time}}$, recovery rate.

Analysis

Let N denote the total population. It follows that

$$N = S + I$$
$$\dot{N} = \dot{S} + \dot{I} = 0$$

$\Rightarrow N$ is constant.

$\Rightarrow S = N - I.$

Therefore,

$$\dot{I} = \alpha(N-I)I - \beta I$$

$\Rightarrow [\alpha] = \frac{1}{\text{pop}\cdot\text{time}}$, $[\beta] = \frac{1}{\text{time}}$, $[N] = \text{population}$

Let

$$x = I/N, \quad \tau = \beta t$$

Which are dimensionless measures of the population of infected and the time scale of recovery. Therefore,

$$\dot{x} = \frac{\dot{I}}{N} = \frac{\alpha}{N}(N-I)I - \frac{\beta}{N} I$$

$$= \frac{\alpha}{N}(N-Nx)Nx - \frac{\beta}{N}Nx$$

$$= \alpha N(1-x)x - \beta x$$

Now,

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \beta \frac{dx}{d\tau}$$

$$\Rightarrow \beta \frac{dx}{d\tau} = \alpha N(1-x)x - \beta x$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{\alpha N}{\beta}(1-x)x - x$$

$$\Rightarrow \frac{dx}{d\tau} = R_0(1-x)x - x.$$

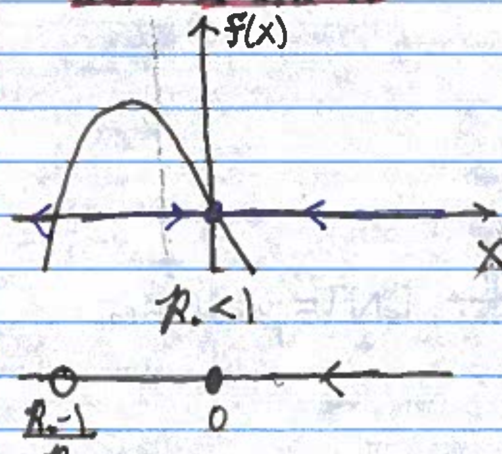
$$\Rightarrow \frac{dx}{d\tau} = x(R_0(1-x)-1) = F(x)$$

Fixed Points

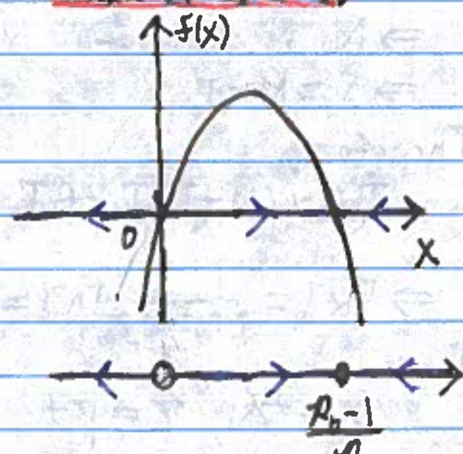
$$x^* = 0, \quad x^* = \frac{R_0 - 1}{R_0}$$

$$\lim_{x\to\infty} F(x) = -\infty \Rightarrow \text{Rightmost fixed point is stable}$$

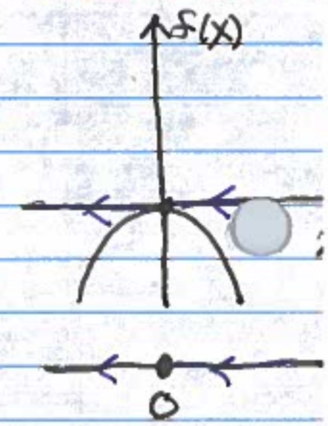
Case 1 $(R_0<1)$



Case 2 $(R_0>1)$

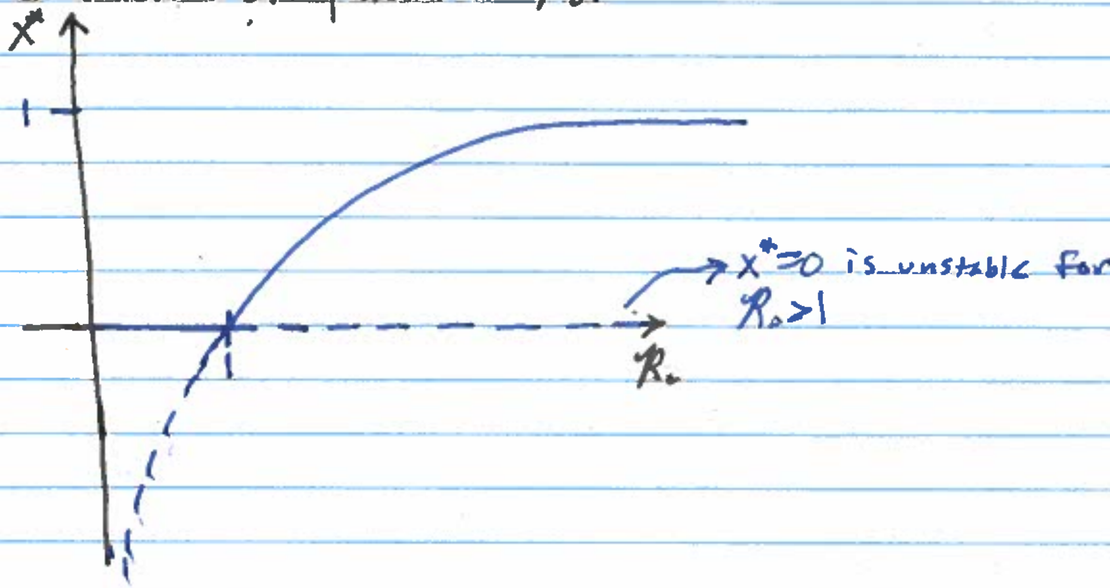


Case 3 $(R_0=1)$



Bifurcation Diagram:

The idea is to plot the location and stability of fixed points as a function of parameter R_0.



$R_0 = 1$ is called a transcritical bifurcation since $x^* = 0$ is fixed for all R_0 but switches stability as other fixed point passes through it