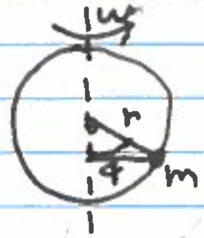
Lecture 8: Bead on a Rotating Hoop



$$mr\ddot{\phi} = \underbrace{-b\dot{\phi}}_{\text{friction}} \underbrace{-mg\sin(\phi)}_{\text{gravity}} + \underbrace{mr\omega^2\sin(\phi)\cos(\phi)}_{\text{centrifugal}}$$

Dimensional Analysis:

$[m] = M$

$[r] = L$

$[mr\ddot{\phi}] = \frac{M\cdot L}{T^2}$

$[b] = \frac{M\cdot L}{T}$

$[g] = \frac{L}{T^2}$

$[\omega] = T^{-1}$

We want to consider "small mass" limit with ω the bifurcation parameter.

Rescale:

$\tau = \frac{t}{T_{sc}}$ ← time scale

$$\Rightarrow \frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{T_{sc}}\frac{d}{d\tau}$$

$$\Rightarrow \frac{mr}{T_{sc}^2}\frac{d^2\phi}{d\tau^2} = \frac{-b}{T_{sc}}\frac{d\phi}{d\tau} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi)$$

Divide by mg to make system dimensionless

$$\frac{r}{gT_{sc}^2}\frac{d^2\phi}{d\tau^2} = \frac{-b}{mg}\frac{d\phi}{d\tau} - \sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi)$$

In order to obtain a first order differential equation we need

$$\varepsilon = \frac{r}{g T_{sc}^2} \ll 1 \,, \quad \frac{b}{m g T_{sc}} = 1$$

$$\Rightarrow T_{sc} = \frac{b}{mg} \Rightarrow \varepsilon = \frac{r}{g(b/mg)^2} = \frac{r m^2 g}{b^2} \ll 1.$$

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma \sin(\phi)\cos(\phi)$$

$$\boxed{\gamma = \frac{r\omega^2}{g}}$$

Since $\varepsilon \ll 1$, the 1-D system is then

$$\frac{d\phi}{d\tau} = \sin(\phi)(\gamma\cos(\phi) - 1)$$

Fixed Points

$\phi = 0, \pi, \cos(\phi) = -1+\gamma$

$$\left.\frac{df}{d\phi}\right|_0 = -1+\gamma \Rightarrow \text{stable if } \gamma < 1$$

$$\left.\frac{df}{d\phi}\right|_\pi = 1+\gamma \Rightarrow \text{always unstable}$$