Homework #1

pg. 12, #3

Suppose $f, g : [a,b] \to \mathbb{R}$ are bounded functions. Prove that

$$L(f, [a,b]) + L(g, [a,b]) \leq L(f+g, [a,b])$$
$$U(f+g, [a,b]) \leq U(f, [a,b]) + U(g, [a,b])$$

proof:

On any interval $[a', b']$ for all $\varepsilon > 0$ there exists $x', y' \in [a', b']$ such that

$$f(x') - \inf_{[a',b']} f(x) < \varepsilon \quad \text{and} \quad g(y') - \inf_{[a',b']} g(x) < \varepsilon$$

and $z' \in [a', b']$ such that

$$f(z') + g(z') - \inf_{[a',b']} (f(x) + g(x)) < \varepsilon.$$

Consequently,

$$\inf_{[a,b]} (f(x) + g(x)) \geq f(z') + g(z') - \varepsilon \geq \inf_{[a,b]} f(x) + \inf_{[a',b']} g(x) - \varepsilon$$

$$\Rightarrow \inf_{[a',b']} f(x) + g(x) \geq \inf_{[a,b]} f(x) + \inf_{[a,b]} g(x).$$

Since this is true for all intervals $[a', b']$ it follows that $L(f, [a,b]) + L(g, [a,b]) \leq L(f+g, [a,b])$. A similar argument proves that $U(f+g, [a,b]) \leq U(f, [a,b]) + U(g, [a,b])$. ■

pg. 7, #3.

Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. Prove that f is Riemann integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f, P, [a,b]) - L(f, P, [a,b]) < \varepsilon.$$

proof:

$(\rightarrow)$ Suppose f is Riemann integrable. Then,

$$\inf_P U(f, P, [a,b]) = U(f, [a,b]) = L(f, [a,b]) = \inf_P L(f, P, [a,b])$$

Consequently for $\varepsilon > 0$ there exists partitions P', P'', such that

$$|L(f, [a,b]) - L(f, P', [a,b])| < \varepsilon/2$$
$$|U(f, P', [a,b]) - U(f, [a,b])| < \varepsilon/2$$

Letting $P''' = P' \cup P''$ it follows that

$$L(f,[a,b]) - L(f,P''',[a,b]) + U(f,P''',[a,b]) - U(f,[a,b]) < \varepsilon$$
$$\Rightarrow U(f,P''',[a,b]) - L(f,P''',[a,b]) < \varepsilon.$$

($\Leftarrow$) Consider a sequence of partitions P_n satisfying

$$U(f,P_n,[a,b]) - L(f,P_n,[a,b]) < 1/n.$$

Therefore,

$$\lim_{n\to\infty} U(f,P_n,[a,b]) = \lim_{n\to\infty} L(f,P_n,[a,b]).$$

Consequently, since

$$L(f,P_n,[a,b]) \le L(f,[a,b]) \le U(f,[a,b]) \le U(f,P_n,[a,b])$$

it follows from squeeze theorem that

$$L(f,[a,b]) = U(f,[a,b]).$$

$\blacksquare$

~~pg. 7, #4~~

Suppose $f,g:[a,b]\to\mathbb{R}$ are Riemann integrable. Prove that $f+g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

~~proof~~:

For any partition P we have from pg. 12, #3:

$$L(f,P,[a,b]) + L(g,P,[a,b]) \le L(f+g,P,[a,b]),$$
$$U(f+g,P,[a,b]) \le U(f,P,[a,b]) + U(g,P,[a,b]).$$

Therefore,

$$\begin{aligned} U(f,[a,b]) + U(g,[a,b]) &= L(f,[a,b]) + L(g,[a,b]) \\ &\le L(f+g,[a,b]) \\ &\le U(f+g,[a,b]) \\ &\le U(f,[a,b]) + U(g,[a,b]) \end{aligned}$$

and $L(f+g,[a,b]) = U(f+g,[a,b])$ proving $f+g$ is Riemann integrable. The same estimate proves that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

$\blacksquare$

pg. 7, #5

Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable. Prove that the function $-f$ is Riemann integrable on $[a,b]$ and

$$\int_a^b(-f) = -\int_a^b f.$$

proof:

For any set A since $\inf(-A) = -\sup(A)$ and $\sup(-A) = -\inf(A)$ it follows that

$$L(-f,P,[a,b]) = -U(f,P,[a,b]),$$
$$U(-f,P,[a,b]) = -L(f,P,[a,b])$$

and thus

$$L(-f,[a,b]) = -U(f,[a,b]) \text{ and } U(-f,[a,b]) = -L(f,[a,b])$$
$$\Rightarrow L(-f,[a,b]) = -L(f,[a,b]) = U(-f,[a,b])$$

proving that $-f$ is Riemann integrable and

$$\int_a^b -f = -\int_a^b f$$

■

pg. 8, #11

Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable. Define $F:[a,b]\to\mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x = a \\ \int_a^x f & \text{if } x \in (a,b] \end{cases}.$$

Prove that F is continuous on $[a,b]$.

proof:

For $x_1, x_2 \in [a,b]$ it follows that if $x_1 < x_2$ then

$$|F(x_2) - F(x_1)| = \left|\int_{x_1}^{x_2} f\right| \le (x_2 - x_1)\sup_{[x_1,x_2]}|f(t)| \le (x_2 - x_1)\sup_{[a,b]}|f(t)|$$

and thus F is Lipschitz continuous.

■

pg. 8, #12

Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable. Prove that $|f|$ is Riemann integrable and

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

proof:

Since f is Riemann integrable, for all $\varepsilon>0$ there exists a partition $P=\{x_0, x_1, \dots, x_n\}$ such that

$$U(f, P, [a,b]) - L(f, P, [a,b]) < \varepsilon.$$

For all $\delta>0$ on each interval $[x_i, x_{i+1}]$ choose x_i^1, x_i^2 so that

$$|f(x_i^1)| - \inf_{[x_i,x_{i+1}]} |f(x)| < \delta \text{ and } \sup_{[x_i,x_{i+1}]} |f(x)| - |f(x_i^2)| < \delta.$$

Therefore,

$$\sup_{[x_i,x_{i+1}]} |f(x)| - \inf_{[x_i,x_{i+1}]} |f(x)| - 2\delta \le |f(x_i^2)| - |f(x_i^1)|.$$

Furthermore, by the reverse triangle inequality we have that

$$|f(x_i^2)| - |f(x_i^1)| \le |f(x_i^2) - f(x_i^1)| = f(x_i^2) - f(x_i^1) \le \sup_{[x_i,x_{i+1}]} f(x) - \inf_{[x_i,x_{i+1}]} f(x).$$

and thus

$$\sup_{[x_i,x_{i+1}]} |f(x)| - \inf_{[x_i,x_{i+1}]} |f(x)| \le \sup_{[x_i,x_{i+1}]} f(x) - \inf_{[x_i,x_{i+1}]} f(x) + 2\delta.$$

Since δ was arbitrary it follows that

$$\sup_{[x_i,x_{i+1}]} |f(x)| - \inf_{[x_i,x_{i+1}]} |f(x)| \le \sup_{[x_i,x_{i+1}]} f(x) - \inf_{[x_i,x_{i+1}]} f(x)$$

and thus

$$U(|f|, P, [a,b]) - L(|f|, P, [a,b]) \le U(f, P, [a,b]) - L(f, P, [a,b]) < \varepsilon$$

proving that $|f|$ is Riemann integrable.

Furthermore, for any partition P it follows that

$$\begin{aligned}
|U(f, P, [a,b])| &= \left|\sum_{i=0}^{n} (x_{i+1}-x_i) \sup_{[x_i,x_{i+1}]} f(x)\right| \\
&\le \sum_{i=0}^{n} (x_{i+1}-x_i) \left|\sup_{[x_i,x_{i+1}]} f(x)\right| \\
&\le \sum_{i=0}^{n} (x_{i+1}-x_i) \sup_{[x_i,x_{i+1}]} |f(x)| \\
&= U(|f|, P, [a,b])
\end{aligned}$$

and thus

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

■

~~pg. 8, #14~~

Suppose $f_n : [a,b] \to \mathbb{R}$ are Riemann integrable and $f_n \to f$ uniformly. Prove that f is Riemann integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

~~proof~~:

Since $f_n \to f$ uniformly for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\inf_{[a,b]} |f(x) - f_n(x)| \leq \sup_{[a,b]} |f(x) - f_n(x)| \leq \varepsilon.$$

Moreover, since f_n are Riemann integrable for each n there exists a partition $P_{n,\varepsilon}$ such that

$$U(f_n\, P_{n,\varepsilon}, [a,b]) - L(f_n, P_{n,\varepsilon}, [a,b]) < \varepsilon.$$

Furthermore,

$$\begin{aligned} \sup_{[a,b]} f - \inf_{[a,b]} f &= \sup_{[a,b]} f - f_n + f_n - \inf_{[a,b]} f - f_n + f_n \\ &\leq \sup_{[a,b]} f - f_n + \sup_{[a,b]} f - \inf_{[a,b]} f - f_n - \inf_{[a,b]} f_n \\ &\leq 2\varepsilon + \sup_{[a,b]} f_n - \inf_{[a,b]} f_n \end{aligned}$$

Therefore,

$$U(f, P_{n,\varepsilon}, [a,b]) - L(f, P_{n,\varepsilon}, [a,b]) \leq 2\varepsilon(b-a) + \varepsilon$$

and thus f is Riemann integrable.

Finally, again choosing $n \geq N$ we have that

$$\begin{aligned} \left| \int_a^b f - \int_a^b f_n \right| &= \left| \int_a^b (f - f_n) \right| \\ &\leq \int_a^b |f - f_n| \\ &\leq \int_a^b \varepsilon \\ &= (b-a)\varepsilon \end{aligned}$$

and thus $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

∎

pg. 7, #2

Suppose $a \leq s < t \leq b$. Define $f:[a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } s < x < t \\ 0, & \text{o.w} \end{cases}$$

Prove that f is Riemann integrable on $[a,b]$ and that

$$\int_a^b f = t - s.$$

proof:

Let P_n be the partition defined by

$$P_n = \{a, s - 1/n, s + 1/n, x_0, \dots, x_n, t - 1/n, t + 1/n, b\}$$

where $x_0, \dots, x_n$ is a uniformly spaced partition of $(s + 1/n, t - 1/n)$ that excludes the initial and terminal points $s + 1/n$ and $t - 1/n$ respectively. Therefore,

$$L(f, P_n, [a,b]) = 2/n \cdot 0 + \sum_{i=0}^{n-1} (x_{i+1} - x_i) \cdot 1 + 2/n \cdot 0 = s - t - 2/n$$

$$U(f, P_n, [a,b]) = 2/n \cdot 1 + \sum_{i=0} (x_{i+1} - x_i) \cdot 1 + 2/n \cdot 1 = s - t + 2/n$$

and consequently,

$$s - t - 2/n = L(f, P_n, [a,b]) \leq L(f, [a,b]) \leq U(f, [a,b]) \leq U(f, P_n, [a,b]) = s - t + 2/n$$

Taking the limit as $n \to \infty$ on both sides we have

$$s - t = L(f, [a,b]) = U(f, [a,b]) = s - t.$$

$\blacksquare$

pg. 12, #5

Give an example of real valued functions $f_1, f_2, \dots$ on $[0,1]$ and a continuous real valued function f on $[0,1]$ such that

$$f(x) = \lim_{k \to \infty} f_k(x)$$

for each $x \in [0,1]$ but

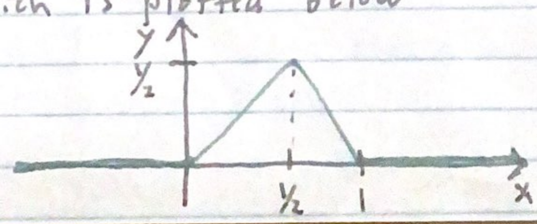
$$\int_0^1 f \neq \lim_{k \to \infty} \int_0^1 f_k.$$

proof:

Let $\bar{f}_1 : \mathbb{R} \to \mathbb{R}$ be defined by:

$$\bar{f}_1(x) = \begin{cases} 1/2 - |x - 1/2|, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

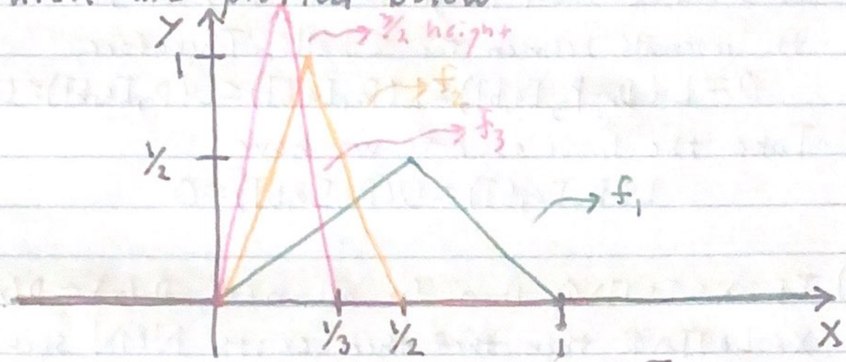
which is plotted below



Define, $\bar{f}_n : \mathbb{R} \to \mathbb{R}$ by

$$\bar{f}_n(x) = n\,\bar{f}_n(nx)$$

which are plotted below



Define, f_n to be the restriction of $\bar{f}_n$ to $[0,1]$, i.e.
$f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \bar{f}_n(x)$.

By construction, $\lim\limits_{n\to\infty} f_n(0) = \lim\limits_{n\to\infty} 0 = 0$ and for $x \in (0,1]$ $\lim\limits_{n\to\infty} f_n(x) = 0$ and thus $\lim\limits_{n\to\infty} f_n(x) = 0$. However,

$$\int_0^1 f_n(x)\,dx = \int_0^1 n f_n(nx)\,dx$$
$$= \int_0^{1/n} n f_n(nx)\,dx.$$

Letting $u = nx$ we have

$$\int_0^1 f_n(x)\,dx = \int_0^1 f_1(x)\,dx = 1.$$

Consequently,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f(x)\,dx.$$

■

<u>Applied Problem #3</u>

Enumerate the rationals in $[0,1]$ as $r_1, r_2, \ldots$, and define

$$D_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, \ldots, r_n\}. \\ 0, & \text{otherwise} \end{cases}$$

(a) Show directly that D_n is Riemann integrable

(b) Prove that $D_n(x)$ converges pointwise to the Dirichlet function $D(x)$.

(c) Show that $D_n(x)$ does not converge uniformly to $D(x)$.

Solution:

(a) Consider a partition P_ε of $r_1, \dots, r_n$ consisting of ε-balls around r_i with ε small enough so that they are not intersecting and the intervals between the ε-balls. Therefore,

$$0 = L(D_n, P_\varepsilon, [a,b]) \le L(D_n, [a,b]) \le U(D_n, [a,b]) \le U(D_n, P_\varepsilon, [a,b]) = 2n\varepsilon.$$

Take the limit as $\varepsilon \to 0$ we have

$$L(D_n, [a,b]) = U(D_n, [a,b]) = 0.$$

(b) If $x \in [0,1] \setminus \mathbb{Q}$ then for all $n \in \mathbb{N}$, $D_n(x) = D(x) = 0$. If $x \in [0,1] \cap \mathbb{Q}$ then then there exists $N \in \mathbb{N}$ such that for all $n \ge N$ implies $D_n(x) = 1 = D(x)$. Consequently, from these two statements it follows that

$$\lim_{n\to\infty} D_n(x) = D(x).$$

(c) For all $n \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}} |D_n(x) - D(x)| = 1$$

and thus D_n does not converge uniformly to D. ∎

Applied Problem #4

Consider the sequence of functions defined by

$$D_n(x) = \cos(n!\pi x)^{2n}.$$

(a) Sketch plots of $D_1, \dots, D_5$.

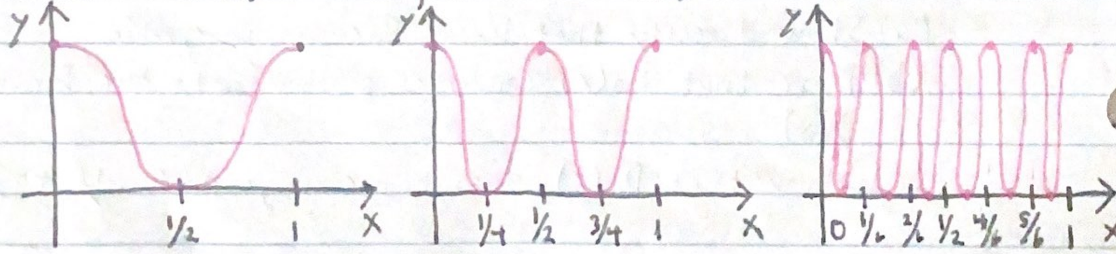
(b) Compute $\int_0^1 D_n(x)\,dx$.

(c) Prove that $D_n(x)$ converges pointwise to the Dirichlet function $D(x)$.

(d) Prove that D_n does not converge uniformly to $D(x)$.

~~proof~~:

$D_1(x) = \cos^2(\pi x)$ $\qquad$ $D_2(x) = \cos(2\pi x)^4$ $\qquad$ $D_3(x) = \cos(6\pi x)^6$



(b). Computing, we have that:

$$\begin{aligned}
\int_0^1 D_n(x)\,dx &= \int_0^1 \cos(n!\pi x)^{2n}\,dx \\
&= \frac{\cos^{2n-1}(n!\pi x)\sin(n!\pi x)}{2n\cdot n!\pi}\Big|_0^1 + \frac{2n-1}{2n}\int_0^1 \cos(n!\pi x)^{2(n-1)}\,dx \\
&= \frac{2n-1}{2n}\int_0^1 D_{n-1}(x)\,dx \\
&= \frac{2n-1}{2n}\,\frac{2(n-1)-1}{2(n-1)}\int_0^1 D_{n-2}(x)\,dx \\
&= \frac{n-\frac{1}{2}}{n}\,\frac{(n-1)-\frac{1}{2}}{n-1}\int_0^1 D_{n-2}(x)\,dx \\
&\ \vdots \\
&= \frac{n-\frac{1}{2}}{n}\,\frac{(n-1)-\frac{1}{2}}{n-1}\,\frac{(n-2)-\frac{1}{2}}{n-2}\cdots\frac{\frac{1}{2}}{1}\int_0^1 \cos^2(n!\pi x)\,dx \\
&= \prod_{i=0}^{n-1}\left(\frac{n-i-\frac{1}{2}}{n-i}\right)\int_0^1 \cos^2(n!\pi x)\,dx \\
&= \prod_{i=0}^{n-1}\left(\frac{n-i-\frac{1}{2}}{n-i}\right)\int_0^1 \frac{1+\cos(2n!\pi x)}{2}\,dx \\
&= \frac{1}{2}\prod_{i=0}^{n-1}\left(\frac{n-i-\frac{1}{2}}{n-i}\right).
\end{aligned}$$

(c) Let $x^* \in [0,1] \sim \mathbb{Q}$. Then there exists $p, q \in \mathbb{N}$ such that

$$x^* = p/q.$$

Consequently, for all $n \geq q$ it follows that

$$n!\pi x^* = (n\cdot(n-1)\cdots(q+1)\cdot q\cdot(q-1)\cdots 1)\pi\cdot\frac{p}{q} = n(n-1)\cdots(q+1)(q-1)\cdots 1\pi p$$

and thus $\cos^{2n}(n!\pi x^*) = \cos^{2n}(n!\pi p/q) = 1$. Therefore,

$$\lim_{n\to\infty} D_n(x) = 1 = D(x).$$

Now, suppose $x^* \in [0,1] \setminus \mathbb{Q}$ and consider the sequence defined by $x_n = \mathrm{mod}(n!\pi x^*, 1)$. Note, $\cos(x_n) = \cos(n!\pi x^*)$ since $\cos(n!\pi x)$ has period 1. By construction, x_n creates a dense orbit in $[0,1]$ and thus does not converge. Consequently, $\cos(n!\pi x_n)$ is bounded away from 1 and thus

$$\lim_{n\to\infty} \cos(n!\pi x_n)^{2n} = 0.$$

(d) Suppose $D_n \to D$ uniformly. Consequently, since D_n are each continuous it follows that D is Riemann integrable which is a contradiction and thus D_n cannot converge to D uniformly. $\blacksquare$