MTH 381
Homework #2

1 Theory Problems

1. pg. 23-24, #1, #3, #11.

2 Applied Problems

1. pg. 23-24, #2, #6, #7, #10.

Homework #2

pg. 23, #1

Prove that if $A, B \subset \mathbb{R}$ and $|B| = 0$, then $|A \cup B| = |A|$.

proof:

Since $A \subset A \cup B$ it follows that $|A| \leq |A \cup B|$. Also, by subadditivity it follows that

$$|A \cup B| \leq |A| + |B| = |A|.$$

$\blacksquare$

pg. 23, #2

Suppose $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. Let $tA = \{ta : a \in A\}$. Prove that $|tA| = |t| \cdot |A|$.

proof:

For any open intervals I_k' such that $tA \subset \bigcup_{k=1}^{\infty} I_k'$ there is a bijection with open intervals I_k such that $A \subset \bigcup_{k=1}^{\infty} I_k$ defined by

$$I_k = \tfrac{1}{t} I_k' \Rightarrow t I_k = I_k'.$$

Consequently, for any open intervals I_k' such that $tA \subset \bigcup_{k=1}^{\infty} I_k'$ we have that

$$\sum_{k=1}^{\infty} \ell(I_k') = \sum_{k=1}^{\infty} |t| \ell(I_k) = |t| \sum_{k=1}^{\infty} \ell(I_k).$$

Taking the inf over both sides we have that

$$|tA| = |t| \cdot |A|.$$

$\blacksquare$

pg. 23, #3

Prove that if $A, B \subset \mathbb{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

proof:

Since $B \subseteq (B \setminus A) \cup A$ it follows that

$$|B| \leq |B \setminus A| + |A|$$

$$\Rightarrow |B \setminus A| \geq |B| - |A|.$$

$\blacksquare$

~~pg.~~ 23, #6

Prove that if $a<b$ then

$$|(a,b)|=|[a,b]|=|(a,b]|=b-a$$

~~proof~~:

By considering the sequence $(a-\varepsilon, b+\varepsilon)$ it follows that

$$|(a,b)|\leq 2\varepsilon+b-a.$$

Furthermore, by considering the closed interval $[a+\varepsilon, b-\varepsilon]$ it follows that $|(a,b)|\geq b-a-2\varepsilon$. Similar arguments show the other intervals satisfy:

$$|(a,b)|=|[a,b)|=|(a,b]|=b-a.$$

■

~~pg.~~ 23, #7

Suppose $a,b,c,d\in\mathbb{R}$ with $a<b$ and $c<d$. Prove that

$$|(a,b)\cup(c,d)|=b-a+d-c$$

if and only if $(a,b)\cap(c,d)=\emptyset$.

~~proof~~:

First suppose $(a,b)\cap(c,d)=\emptyset$. Then,

$$|(a,b)\cup(c,d)|\leq|(a,b)|+|(c,d)|=b-a+d-c.$$

Moreover,

$$|(a,b)\cup(c,d)|=|(a,d)\setminus[b,c]|\geq|(a,d)|-(b,c)=d-a+c-d.$$

Now, if $(a,b)\cap(c,d)\neq\emptyset$ then $(a,b)\cup(c,d)$ is a connected open interval and thus

$$|(a,b)\cup(c,d)|=\max\{b,d\}-\min\{a,c\}\neq d-c+b-a.$$

■

#10.

Prove that $|[0,1]\setminus\mathbb{Q}|=1$.

proof:

Since $[0,1]\setminus\mathbb{Q}\subseteq[0,1]$ it follows that

$$|[0,1]\setminus\mathbb{Q}|\le|[0,1]|=1.$$

We also have that

$$|[0,1]\setminus\mathbb{Q}|\ge|[0,1]|-|[0,1]\cap\mathbb{Q}|=|[0,1]|=1.$$

■

#11

Prove that if $I_1, I_2, \ldots$ is a disjoint sequence of open intervals, then

$$\left|\bigcup_{k=1}^{\infty} I_k\right|=\sum_{k=1}^{\infty}\ell(I_k).$$

proof:

By subadditivity it follows that

$$\left|\bigcup_{k=1}^{\infty} I_k\right|\le\sum_{k=1}^{\infty}|I_k|=\sum_{k=1}^{\infty}\ell(I_k).$$

Now, by problem #2 it follows that

$$\sum_{k=1}^{n}|I_k|=\left|\bigcup_{k=1}^{n} I_k\right|\le\left|\bigcup_{k=1}^{\infty} I_k\right|\le\sum_{k=1}^{\infty}\ell(I_k).$$

The result follows by an application of squeeze theorem.