MTH 381
Homework #4

Due Date: September 23, 2022

1 Theory Problems

1. pg. 38-39, #13, #14, #16, #21

2 Applied Problems

1. pg. 38, #9
2. There are many equivalent definitions of measurability of real-valued functions on a measure space. Prove that each of the following properties implies that f is measurable, and that a measurable $f : X \mapsto \mathbb{R}$ satisfies each condition.
   (a) For every $\alpha \in \mathbb{R}$, $\{x : f(x) \leq \alpha\} \in \mathcal{B}$.
   (b) For every $\alpha \in \mathbb{R}$, $\{x : f(x) > \alpha\} \in \mathcal{B}$.
   (c) For every $\alpha \in \mathbb{R}$, $\{x : f(x) \geq \alpha\} \in \mathcal{B}$.
   (d) For every interval $(a, b) \subset \mathbb{R}$, $\{x : f(x) \in (a, b)\} \in \mathcal{B}$.
3. The set
$$\mathbb{R}_{\text{ex}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$$
   is called the **extended real numbers**. It is subject to the following rules of arithmetic:
   (a) if $a \in \mathbb{R}$ then $a \pm \infty = \pm\infty$,
   (b) if $a > 0$, then $a \times \pm\infty = \pm\infty$,
   (c) $0 \times \pm\infty = 0$,
   (d) $\infty - \infty$ is undefined. If $(X, \mathcal{B})$ is a measurable space.

   A function $f : X \mapsto \mathbb{R}_{\text{ex}}$ is said to be **measurable** if the set $\{t : f(t) < \alpha\}$ belongs to $\mathcal{B}$ for each $\alpha \in \mathbb{R}$.

   Let $f : X \mapsto \mathbb{R}_{\text{ex}}$ be measurable. Show that the sets $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ are measurable.
4. Let X be a set and let $B_1, \ldots B_N$ be disjoint sets whose union is X. Let β be the smallest σ-algebra containing all B_j.
   (a) Describe β by listing all the sets it contains.
   (b) Describe all measurable functions as explicitly as possible.