MTH 381
Homework #6

Due Date: October 21, 2022

1 Theory Problems

1. pg. 84-85: #1, #3, #9, #15.
2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a positive Lebesgue measurable function. Prove that
$$\mu(\{x \in \mathbb{R} : f(x) > \varepsilon\}) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}} f(x) d\mu.$$
3. Consider a family $\{x_{n,\alpha}\}$ of real numbers indexed by $n \in \mathbb{N}$ and $\alpha \in A$. Prove that
$$\sup_{\alpha \in A} \left(\liminf_{n \to \infty} x_{n,\alpha} \right) \leq \liminf_{n \to \infty} \left(\sup_{\alpha \in A} x_{n,\alpha} \right).$$

2 Applied Problems

1. pg. 84-85: #2, #8, #13, #14.
2. Prove or disprove: if
$$\int_{\mathbb{R}} |f(x)| dx < \infty$$
then $\lim_{x \to \infty} f(x) = 0$.
3. Let $f_n \geq 0$ be measurable. Prove that
$$\int_{\mathbb{R}} \sum_{n=0}^{\infty} f_n(x) d\mu = \sum_{n=0}^{\infty} \int_{\mathbb{R}} f_n(x) d\mu.$$
4. Prove that
$$\lim_{n \to \infty} \int_0^{\pi} (1 - (\sin(x))^n) dx = \pi$$
and use this to deduce that
$$\lim_{n \to \infty} \int_0^{\pi} (\sin(x))^n dx = 0.$$
5. Find, with proof, a series expansion for the definite integral
$$\int_0^1 \frac{x^a}{1 + x^b} dx,$$
where $a, b > 0$. **Hint:** Look up the geometric series, i.e. the Taylor series for $1/(1-x)$, and think about how to use monotone convergence theorem.