Lecture 1: Convergence of Functions

Definition - A sequence of functions $f_n: I \to \mathbb{R}$ converges pointwise to f if for all $x \in I$

$$\lim_{n\to\infty} f_n(x) = f(x) \iff \lim_{n\to\infty} |f_n(x) - f(x)| = 0.$$

Definition - A sequence of functions $f_n: I \to \mathbb{R}$ converges uniformly to f if

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Example:

Let $f_n: [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$. It follows that $f_n \to f$ pointwise but not uniformly for

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$$

proof:

For $x \in [0,1)$ we have that $\lim_{n\to\infty} x^n = 0$. If $x = 1$ then for all $n \in \mathbb{N}$ $x^n = 1$ and thus $\lim_{n\to\infty} x^n = 1$.

To prove the lack of uniform convergence we will use a theorem.

Theorem - If a sequence of continuous functions $f_n: I \to \mathbb{R}$ converges uniformly to f then f is continuous.

proof:

First note that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \end{aligned}$$

Since $f_n \to f$ uniformly, for all $\varepsilon > 0$ we can choose $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \varepsilon$, $|f_N(y) - f(y)| < \varepsilon$ for all $x, y \in I$. Furthermore, we can choose $\delta > 0$ such that $|x - y| < \delta$ implies $|f_N(x) - f_N(y)| < \varepsilon$ and thus $|x-y| < \delta$ implies

$$|f(x) - f(y)| \leq 3\varepsilon. \quad \blacksquare$$