MTH 317/617
Fall 2023
Exam 1
09/29/23

Name (Print): Key

This exam contains 11 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Resources:** You are allowed to use your text, notes, and a calculator. You are not allowed to use the internet, software or any other external resource.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Undergraduate Problems:** Questions labeled as **"Undergraduate problems"** will only count for undergraduate students. Graduate students do not have to complete these problems.
- **Graduate Problems:** Questions labeled as **"Graduate problems"** must be completed by the graduate students to receive credit. Undergraduate students can complete these problems for extra credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	10	
7	5	
8	5	
9	10	
10	0	
11	10	
12	0	
Total:	100	

Do not write in the table to the right.

1. (10 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If $z_1, z_2 \in \mathbb{C}$ then $\text{Im}(z_1 z_2) = \text{Im}(z_1)\text{Im}(z_2)$.

   C **I** If $z_1, z_2 \in \mathbb{C}$ then $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

   C **I** If $z \in \mathbb{C}$ satisfies $e^z = -1$ then $z = i\pi$.

   C **I** $i < 10i$.

   C **I** If $z \in \mathbb{C}$ satisfies $|z - 2| < 2$ then $|\text{Arg}(z)| \leq \pi/2$.

2. (10 points) Convert the following complex numbers $z \in \mathbb{C}$ into exponential form, i.e. $z = re^{i\theta}$.

   (a) (5 points) $z = -\sqrt{2}i$

   $-\sqrt{2}i = \sqrt{2}e^{-i\pi/2}$

   (b) (5 points) $z = 1 - i$

   $1 - i = \sqrt{2}e^{-i\pi/4}$

3. (10 points) (**Short Answer**) Match the following complex numbers with the points drawn below on the complex plane $\mathbb{C}$. If no points drawn correspond to the value of z, simply write **no solution**.

(a) (2.5 points) $z = 2i$

O

(b) (2.5 points) $z = -1 - i$

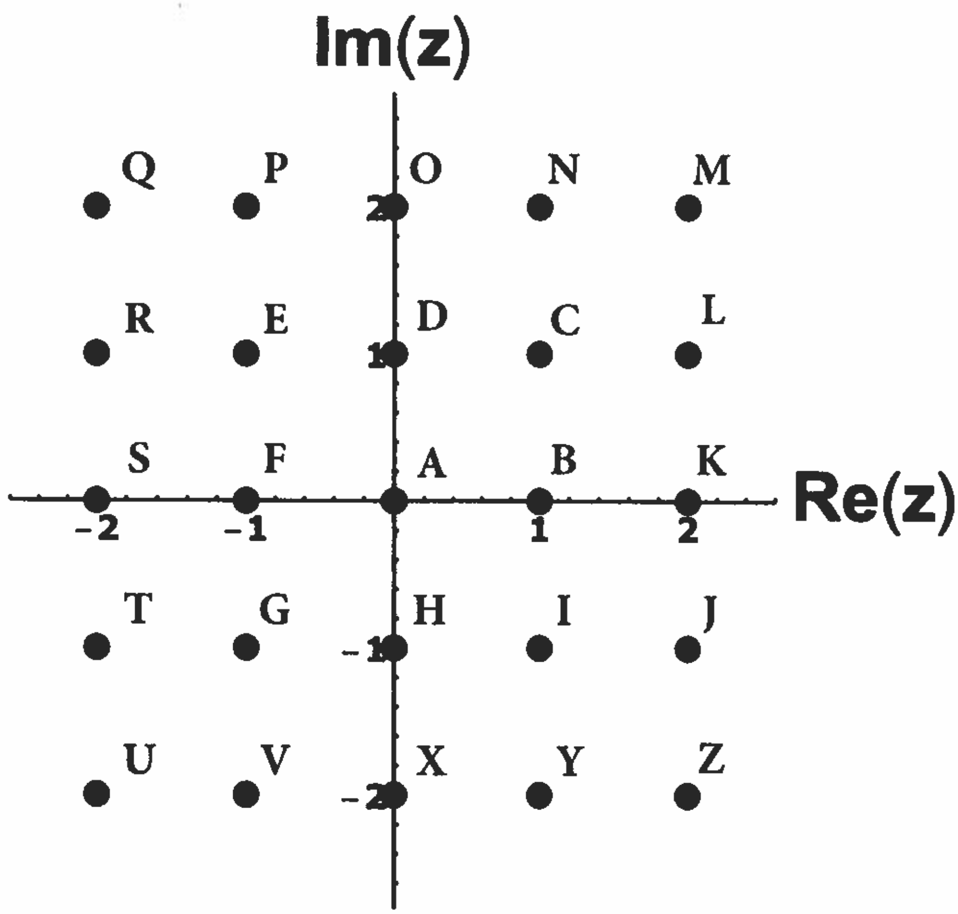
G

(c) (2.5 points) $z = -2\sqrt{2}e^{\frac{3\pi i}{4}}$

Z

(d) (2.5 points) $z = e^0$

B



4. (15 points) Find all solutions to the following equations for $z \in \mathbb{C}$. You can leave your solutions in Cartesian or exponential form.

   (a) (5 points) $\dfrac{1+z}{z} = 1 + 2i$

   $\Rightarrow 1 + z = z + 2iz$

   $\Rightarrow z = \dfrac{1}{2i} = -\dfrac{i}{2}.$

   (b) (10 points) $z^3 = 8e^{\frac{i\pi}{4}}$.

   $\Rightarrow z^3 = 8e^{i\pi/4 + 2n\pi i}$

   $\Rightarrow z = 2e^{i\pi/8 + 2n\pi i/3}$

   for $n \in \mathbb{Z}$.

5. (15 points) (**Short Answer**) For each of the following sets, circle which statements are true. Then, determine the boundary of the set. You just need to list your results, no explanation is needed.

(a) (5 points) $A = \{z \in \mathbb{C} : 2 \leq |z - i| \leq 3\}$

- A is open
- A is closed
- A is connected
- A is bounded

Boundary of A:

$\{z : |z| = 2 \text{ or } |z| = 3\}$.

(b) (5 points) $B = \{z \in \mathbb{C} : \mathrm{Re}(z^2) > 0\}$

- B is open
- B is closed
- B is connected
- B is bounded

Boundary of B:

$\{z = x + iy \in \mathbb{C} : y = \pm x\}$.

(c) (5 points) $C = \{z \in \mathbb{C} : |z| \neq 5\}$

- C is open
- C is closed
- C is connected
- C is bounded

Boundary of C:

$\{z \in \mathbb{C} : z = 5\}$.

6. (10 points) (**Short Answer**) Let $S = \{z \in \mathbb{C} : |z| \leq 3 \text{ and } \mathrm{Re}(z) > 0\}$.

(a) (5 points) If $f(z) = iz$, then the image of S under f is which of the following sets (just circle your answer):

- $\{z \in \mathbb{C} : |z| \leq 3 \text{ and } \mathrm{Re}(z) > 0\}$
- $\{z \in \mathbb{C} : |z| \leq 3 \text{ and } \mathrm{Re}(z) < 0\}$
- $\{z \in \mathbb{C} : |z| \leq 3 \text{ and } \mathrm{Im}(z) > 0\}$
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- $\{z \in \mathbb{C} : |z| \geq 3 \text{ and } \mathrm{Im}(z) < 0\}$

(b) (5 points) If $f(z) = z^{-1}$, then the image of S under f is which of the following sets (just circle your answer):

- $\{z \in \mathbb{C} : |z| \leq 1/3 \text{ and } \mathrm{Re}(z) > 0\}$
- $\{z \in \mathbb{C} : |z| \leq 1/3 \text{ and } \mathrm{Re}(z) < 0\}$
- $\{z \in \mathbb{C} : |z| \leq 1/3 \text{ and } \mathrm{Im}(z) > 0\}$
- $\{z \in \mathbb{C} : |z| \leq 1/3 \text{ and } \mathrm{Im}(z) < 0\}$
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- $\{z \in \mathbb{C} : |z| \geq 1/3 \text{ and } \mathrm{Im}(z) > 0\}$
- $\{z \in \mathbb{C} : |z| \geq 1/3 \text{ and } \mathrm{Im}(z) < 0\}$

7. (5 points) Write down Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

8. (5 points) Write the function $f(z) = \dfrac{z+i}{z-i}$ in the form $w = u(x,y) + iv(x,y)$.

$$\frac{z+i}{z-i} = \frac{z+i}{z-i} \cdot \frac{\bar{z}+i}{\bar{z}+i}$$

$$= \frac{z\bar{z} + \bar{z}i + zi - 1}{|z|^2+1}$$

$$= \frac{|z|^2 + i(\bar{z}+z) - 1}{|z|^2+1}$$

$$= \frac{|z|^2 + 2i\,\mathrm{Re}(z) - 1}{|z|^2+1}$$

$$= \frac{x^2+y^2-1}{x^2+y^2+1} + i\,\frac{2x}{x^2+y^2+1}$$

9. (10 points) (**Undergraduate Problem**) Do either part (a) or part (b). You only need to do one problem to receive full credit. If you attempt both part (a) and part (b) circle or otherwise indicate which problem you want graded. You will not receive extra points if you do both problems.

   (a) (5 points) Prove that for all $\theta_1, \theta_2 \in [0, 2\pi)$,
   $$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

   (b) (5 points) Prove that for all $\theta \in [0, 2\pi)$,
   $$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

a.) $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2).$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$\Rightarrow \sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2$

(b) $(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$

$(e^{i\theta})^2 = e^{2i\theta}$

$= \cos 2\theta + i\sin 2\theta$

$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta.$

10. **(Graduate Problem)** Prove that if $z = \cos(\theta) + i\sin(\theta)$ then

$$\frac{z^2-1}{z^2+1} = i\tan(\theta).$$

$$\frac{z^2-1}{z^2+1} = \frac{e^{2i\theta}-1}{e^{2i\theta}+1}$$

$$= \frac{e^{i\theta}(e^{i\theta}-e^{-i\theta})}{e^{i\theta}(e^{i\theta}+e^{-i\theta})}$$

$$= \frac{2i\sin\theta}{2\cos\theta}$$

$$= i\tan\theta.$$

11. (10 points) (**Undergraduate Problem**) Do either part (a) or part (b). You only need to do one problem to receive full credit. If you attempt both part (a) and part (b) circle or otherwise indicate which problem you want graded. You will not receive extra points if you do both problems.

   (a) (10 points) Consider the sequence $z_n \in \mathbb{C}$ defined by

$$z_n = \frac{e^{in\pi} + in}{n}.$$

   Prove that

$$\lim_{n\to\infty} z_n = i.$$

   (b) (10 points) Suppose that $z_n \in \mathbb{C}$ satisfies $z_n \to 0$ as $n \to \infty$ and $z_n \neq 0$. Prove that

$$\lim_{n\to\infty} z_n e^{i\cos(|z_n|)} = 0.$$

(a) $$|z_n - i| = \left|\frac{e^{in\pi} + in}{n} - i\right| = \left|\frac{e^{in\pi}}{n}\right| = 1/n$$

Therefore,

$$\lim_{n\to\infty} |z_n - i| = 0$$

and thus

$$\lim_{n\to\infty} z_n = i.$$

(b). $$|z_n e^{i\cos(|z_n|)}| = |z_n|$$

and thus $\lim_{n\to\infty} |z_n e^{i\cos(|z_n|)}| = 0$. Therefore,

$$\lim_{n\to\infty} z_n e^{i\cos(|z_n|)} = 0.$$