**MTH 317/617**
**Fall 2023**
**Exam 2**
**11/03/23**

**Name (Print):** Key.

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This exam contains 10 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Undergraduate Problems:** Questions labeled as **"Undergraduate problems"** will only count for undergraduate students. Graduate students do not have to complete these problems.
- **Graduate Problems:** Questions labeled as **"Graduate problems"** must be completed by the graduate students to receive credit. Undergraduate students can complete these problems for extra credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 0 | |
| 8 | 15 | |
| 9 | 10 | |
| 10 | 0 | |
| Total: | 100 | |

Do not write in the table to the right.

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   **C** **I** If $f(z) = u(x,y) + iv(x,y)$ is analytic then $g(z) = v(x,y) - iu(x,y)$ is analytic.

   **C** **I** For all nonzero complex numbers $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$.

   **C** **I** If $z$ is a nonzero complex number then $\text{Log}(z^{-1}) = -\text{Log}(z)$.

   **C** **I** For all $z \in \mathbb{C}$, $|\sin(z)| \leq 1$.

   **C** **I** If $C_1$ and $C_2$ are any closed contours in $\mathbb{C}$ then

$$\int_{C_1} \frac{1}{z-i} dz = \int_{C_2} \frac{1}{z-i} dz.$$

2. (15 points)

   (a) (5 points) **Short Answer:** State the Cauchy Riemann equations which must be satisfied by an analytic function $f(z) = u(x,y) + iv(x,y)$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   (b) (10 points) Find constants $a$ and $b$ such so that the following function is analytic

$$f(z) = 4x^2 + 5x - 4y^2 + 9 + i(axy + by - 1).$$

$$\frac{\partial u}{\partial x} = 8x + 5 = \frac{\partial v}{\partial y} = ax + b$$

$$\frac{\partial u}{\partial y} = -8y = -\frac{\partial v}{\partial x} = -a\cancel{x}y$$

$\Rightarrow a = 8$

$\Rightarrow b = 5$

3. (15 points) Express all of the following in the form $x + iy$ where $x, y \in \mathbb{R}$. In each problem assume the principal branch of Log.

   (a) (5 points) $\text{Log}(3+3i)$

$$\text{Log}(3+3i) = \text{Log}(3\sqrt{2}e^{i\pi/4})$$
$$= \ln(3\sqrt{2}) + i\pi/4$$

   (b) (5 points) $\text{Log}\left(\frac{1}{2}e^{\frac{3}{2}\pi i}\right) = \ln(1/2) - \frac{i\pi}{2}.$

   (c) (5 points) $(1+i)^{1-i}$

$$(1+i)^{1-i} = e^{(1-i)\text{Log}(1+i)}$$
$$= e^{(1-i)(\ln(\sqrt{2}) + i\pi/4)}$$
$$= e^{\ln(\sqrt{2}) + \pi/4} e^{i(\pi/4 - \ln(\sqrt{2}))}$$
$$= \sqrt{2}\, e^{\pi/4}\left(\cos(\pi/4 - \ln(\sqrt{2})) + i\sin(\pi/4 - \ln(\sqrt{2})\right).$$

4. (10 points) Determine the domain of analyticity for the function $f(z) = \mathrm{Log}(2z+i)$ and sketch the branch cuts on the complex plane.

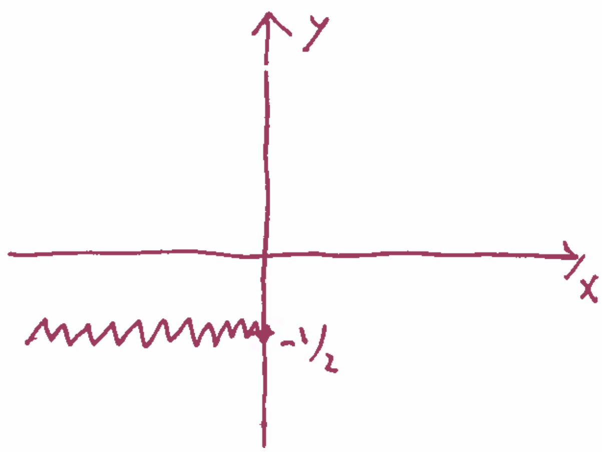
$f(z)$ is not analytic when

$\mathrm{Im}(2z+i) = 0$ and $\mathrm{Re}(2z+i) \le 0$

$\Rightarrow 2y+1=0$ and $x \le 0$

$\Rightarrow y = -1/2$ and $x \le 0$.

The branch cut is illustrated below



5. (10 points) Let $f(z) = \sin(z)$.

(a) (5 points) Write down the exponential form of $\sin(z)$.

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

(b) (5 points) Show that $\overline{f(\overline{z})} = -f(z)$.

$$\overline{f(\overline{z})} = \overline{\frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i}}$$

$$= \frac{e^{-iz} - e^{+iz}}{-2i}$$

$$= \sin(z)$$

$$= f(z).$$

6. (10 points) **(Undergraduate Problem:)** Show that there does not exist an analytic function $f(z) = u(x,y) + iv(x,y)$ for which

$$u(x,y) = y^3 + 5x.$$

Computing we have that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y \neq 0.$$

Since $u$ is not harmonic it follows that $f$ cannot be analytic.

7. (0 points) **(Graduate Problem:)** Suppose $f : \mathbb{C} \mapsto \mathbb{C}$ defined by $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is analytic in a domain $D$. Determine if the function

$$g(z) = \overline{f(\overline{z})}$$

is an analytic function.

$$g(z) = u(x,-y) - i\,v(x,-y).$$
$$= \tilde{u}(x,y) + i\,\tilde{v}(x,y)$$

$$\Rightarrow \frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x}\Big|_{(x,-y)}, \quad \frac{\partial \tilde{v}}{\partial y} = \frac{\partial v}{\partial y}\Big|_{(x,-y)} = \frac{\partial u}{\partial x}\Big|_{(x,-y)}$$

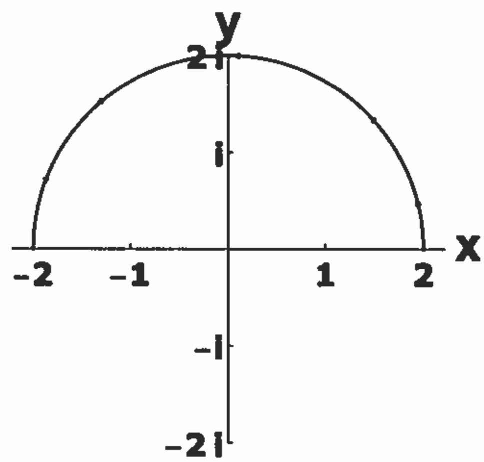
$$\Rightarrow \frac{\partial \tilde{u}}{\partial y} = -\frac{\partial u}{\partial x}\Big|_{(x,-y)}, \quad \frac{\partial \tilde{v}}{\partial x} = -\frac{\partial v}{\partial x}\Big|_{(x,-y)} = \frac{\partial u}{\partial x}\Big|_{(x,-y)}$$

Therefore,

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{v}}{\partial y} \text{ and } \frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{v}}{\partial x}$$

and thus $g(z)$ is analytic.

8. (15 points) Let $C$ be the semicircle connecting $z = 2$ to $z = -2$ oriented in the positive sense; see the below figure.




(a) (5 points) Write down a parametrization of this contour. Be sure to include your domain of parametrization.

$z = 2e^{it}, \quad t \in [0, \pi]$

(b) (5 points) Compute the following contour integral: $\int_C z^4 dz$.

$\int_C z^4 dz = \frac{z^5}{5}\Big|_2^{-2} = -\frac{64}{5}$

(c) (5 points) Compute the following contour integral: $\int_C |z|^2 dz$.

$\int_C |z|^2 dz = \int_0^{\pi} 4(-ie^{-it}) \cdot 2\,dt$

$= 8e^{-it}\Big|_0^{\pi}$

$= -16$

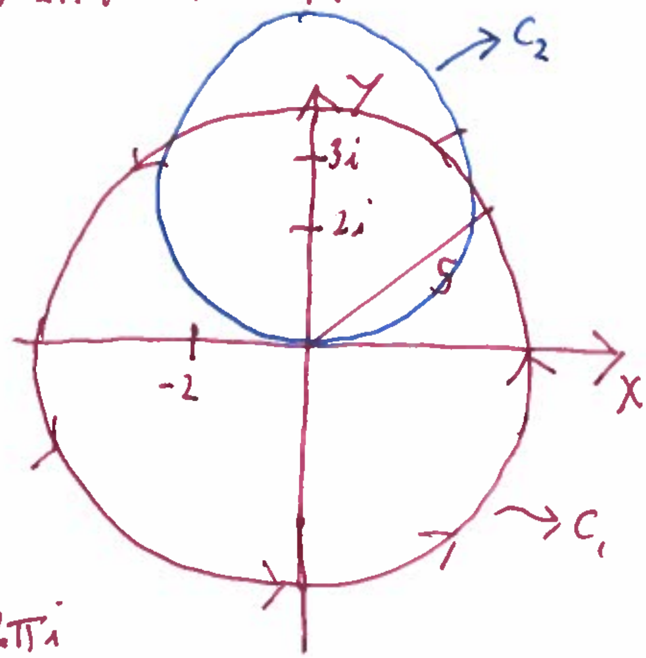
9. (10 points) **(Undergaduate Problem:)** Let $f(z)$ be defined by

$$f(z) = \frac{3}{z+2} - \frac{1}{z-2i},$$

$C_1$ be the closed contour satisfying $|z| = 5$, and $C_2$ be the closed contour satisfying $|z - 3i| = 3$.

(a) (5 points) Compute $\int_{C_1} f(z)dz$.

$$\int_{C_1} f(z)dz = 3 \cdot 2\pi i - 2\pi i = 4\pi i$$



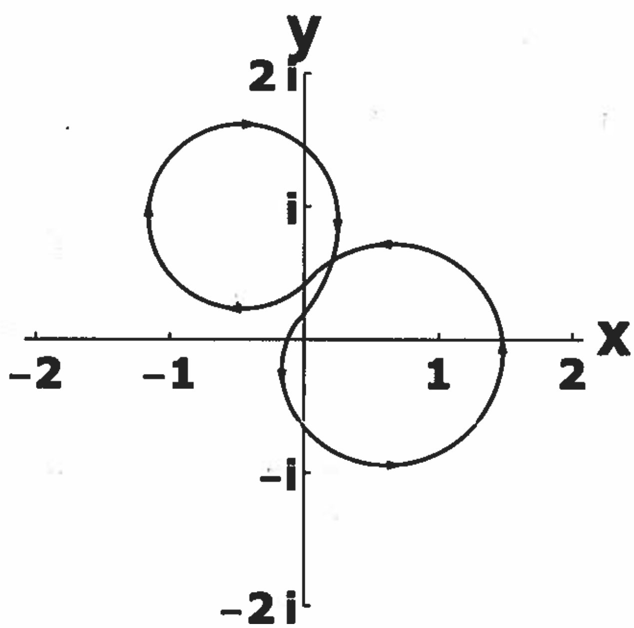

(b) (5 points) Compute $\int_{C_2} f(z)dz$.

$$\int_{C_2} f(z)dz = -2\pi i$$

10. (10 points) **(Graduate Problem:)** Evaluate the integral

$$\int_C \frac{z}{(z-i)(z-1)}dz,$$

where $C$ is the figure-eight path shown below.




$$\int_C \frac{z}{(z-i)(z-1)}dz = \frac{2\pi i \cdot}{1-i} - \frac{2\pi i \cdot i}{i-1}$$

$$= 2\pi i \frac{(1+i)}{(1-i)}$$