MTH 317/617
Homework #2

Due Date: September 15, 2023

1 Problems for Everyone

1. Sketch the following curves in the complex plane:
   (a) $z(t) = 3e^{it}, 0 \leq t \leq 2\pi$.
   (b) $z(t) = 2e^{-it}, 0 \leq t \leq \pi$.
   (c) $z(t) = e^{-(1+i)t}, 0 \leq t \leq 2\pi$.
   (d) $z(t) = e^{(1+i)t}, 0 \leq t \leq 2\pi$.

2. Find all the values of the following. Express your answers in the form $z = a+ib$ where $a, b \in \mathbb{R}$.
   (a) $(-16)^{1/4}$.
   (b) $i^{1/4}$.
   (c) $(1-\sqrt{3}i)^{1/3}$.
   (d) $\left(\frac{2i}{1+i}\right)^{1/6}$.
   (e) $i^{\sqrt{2}}$.
   (f) $(\sqrt{2})^i$.

3. Find all solutions $z \in \mathbb{C}$ to the following equation
$$(z+2)^5 + z^5 = 0.$$

4. Let $a_0, \ldots, a_n \in \mathbb{R}$ and assume $z^* \in \mathbb{C}$ is a root of the polynomial
$$p(z) = a_0 + a_1 z + \ldots a_n z^n.$$
   (a) Prove that for all $z \in \mathbb{C}$, $p(\overline{z}) = \overline{p(z)}$.
   (b) Prove that $\overline{z^*}$ is also a root of the polynomial.

5. Prove that for all $n \in \mathbb{N}$,
$$\left(\frac{1+i\tan(\theta)}{1-i\tan(\theta)}\right)^n = \frac{1+i\tan(n\theta)}{1-i\tan(n\theta)}.$$

6. Prove the following identity:
$$\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta).$$

7. A point z_0 in a set D is called an interior point of D if there some neighborhood centered at z_0 contained in D. Hence, D is open if and only if it contains all of its interior points.

   For each of the following sets in $\mathbb{C}$, (a) sketch the set in $\mathbb{C}$, (b) describe the interior and the boundary, (c) state whether the set is open or closed, (d) state whether the interior is connected, (e) state if the set is bounded or unbounded.

   (a) $A = \{z = x + iy \in \mathbb{C} : x \geq 2 \text{ and } y \leq 4\}$
   (b) $B = \{z \in \mathbb{C} : |z| < 1 \text{ or } |z - 3| \leq 1\}$
   (c) $C = \{z = x + iy \in \mathbb{C} : x^2 < y\}$
   (d) $D = \{z \in \mathbb{C} : \text{Re}(z^2) = 4\}$
   (e) $E = \{z \in \mathbb{C} : z\bar{z} - 2 \geq 0\}$
   (f) $F = \{z \in \mathbb{C} : z^3 - 2z^2 + 5z - 4 = 0\}$
   (g) $G = \{z = x + iy \in \mathbb{C} : -\pi \leq y < \pi\}$

2 Graduate Problems

1. Prove the following statements.

   (a) The boundary of any set $D \subseteq \mathbb{C}$ is a closed set.
   (b) For $D \subseteq \mathbb{C}$, show that if $p \in D$, then p is either an interior point of D or a boundary point of D.
   (c) Show that a set $D \subseteq \mathbb{C}$ coincides with its boundary if and only if D is closed and D has no interior points.
   (d) Show that if $D \subseteq \mathbb{C}$ and $E \subseteq \mathbb{C}$ is a closed set containing D, then E must contain the boundary of D.
   (e) Show that if $D \subseteq \mathbb{C}$ and $S \subseteq \mathbb{C}$ is an open set that is subset of D, then S must be composed entirely of interior points of D.

Homework #2.

#1

Sketch the following curves in $\mathbb{C}$.

(a) $z(t) = 3e^{it}, \ 0 \leq t \leq 2\pi$

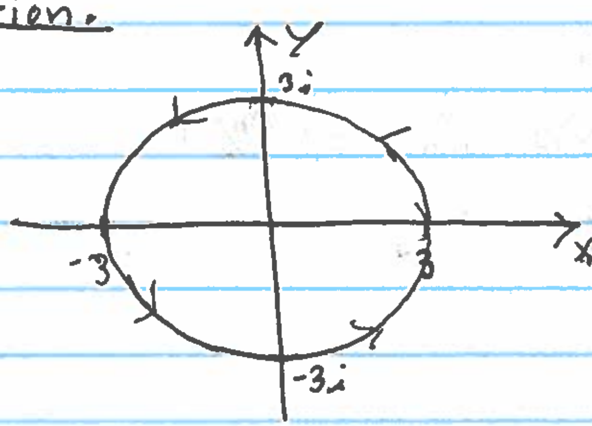
(b) $z(t) = 2e^{-it}, \ 0 \leq t \leq \pi$

(c) $z(t) = e^{-(1+i)t}, \ 0 \leq t \leq 2\pi$

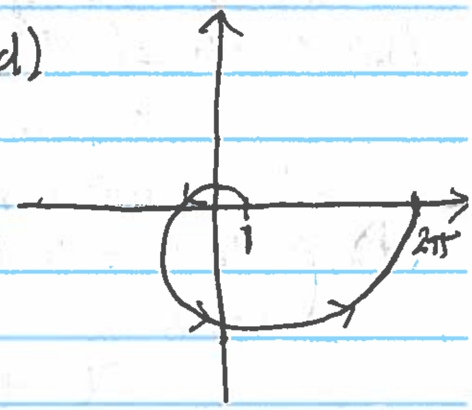
(d) $z(t) = e^{(1+i)t}, \ 0 \leq t \leq 2\pi$

Solution:

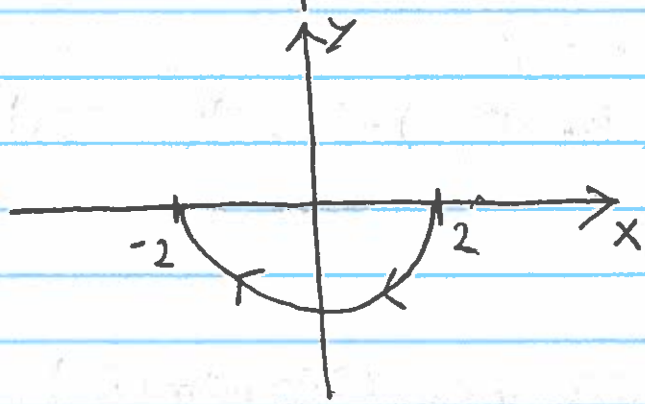
(a)



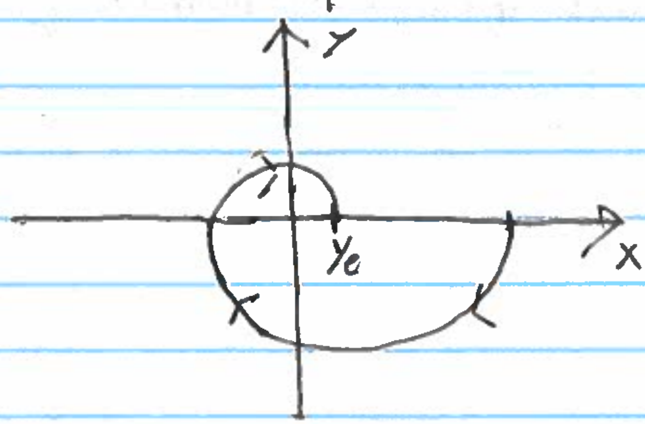
(d)



(b)



(c)



#2.

Find all the values of the following. Express your answers in the form $z = a + ib$, where $a, b \in \mathbb{R}$.

Solution:

(a) $(-16)^{1/4} = \left(16 e^{i\pi + 2\pi i n}\right)^{1/4}$

$$= 2 e^{i\pi/4 + \pi i n/2},$$

$$= 2\left(\cos(\pi/4 + \pi n/2) + i \sin(\pi/8 + \pi n/2)\right),$$

where $n \in \{0, 1, 2, 3\}$.

(b) $(i)^{1/4} = \left(e^{i\pi/2 + 2\pi i n}\right)^{1/4}$

$$= \cos(\pi/8 + \pi n/2) + i \sin(\pi/8 + \pi n/2),$$

where $n \in \{0, 1, 2, 3\}$.

(d) $\left(\dfrac{2i}{1+i}\right)^{1/6} = \left(i(1-i)\right)^{1/6}$

$$= (1 + i)^{1/6}$$

$$= \left(\sqrt{2}\, e^{i\pi/4 + 2\pi i n}\right)^{1/6}$$

$$= 2^{1/12}\left(\cos(\pi/24 + \pi n/3) + i \sin(\pi/24 + \pi n/3)\right),$$

where $n \in \{0, 1, 2, 3\}$.

(e). $i^{\sqrt{2}} = \left(e^{i\pi/2 + 2\pi i n}\right)^{\sqrt{2}}$

$$= e^{i\sqrt{2}\pi/2 + 2\pi i n\sqrt{2}}$$

$$= \cos(\pi/\sqrt{2} + 2\pi\sqrt{2}n) + i \sin(\pi/\sqrt{2} + 2\pi\sqrt{2}n),$$

where $n \in \mathbb{Z}$.

$\blacksquare$

#5.

Prove that for all $n \in \mathbb{N}$,

$$\left(\frac{1+i\tan(\theta)}{1-i\tan(\theta)}\right)^n = \frac{1+i\tan(n\theta)}{1-i\tan(n\theta)}$$

~~proof~~:

$$\left(\frac{1+i\tan(\theta)}{1-i\tan(\theta)}\right)^n = \frac{(\cos\theta + i\sin\theta)^n}{(\cos\theta - i\sin\theta)^n}$$

$$= \frac{\cos(n\theta)+i\sin(n\theta)}{(\cos(-\theta)+i\sin(-\theta))^n}$$

$$= \frac{\cos(n\theta)+i\sin(n\theta)}{\cos(-n\theta)+i\sin(-n\theta)}$$

$$= \frac{\cos(n\theta)+i\sin(n\theta)}{\cos(n\theta)-i\sin(n\theta)}$$

$$= \frac{1+i\tan(n\theta)}{1-i\tan(n\theta)}.$$

■

#6.

Computing we have that

$$e^{2i\theta} = \cos(2\theta)+i\sin(2\theta)$$

$$e^{2i\theta} = (e^{i\theta})^2$$

$$= (\cos(\theta)+i\sin(\theta))^2$$

$$= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$

Therefore,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \text{ and } \sin 2\theta = 2\cos\theta\sin\theta$$

and thus

$$\sin(4\theta) = 2\cos(2\theta)\sin(2\theta)$$

$$= 2(\cos^2\theta - \sin^2\theta)2\cos\theta\sin\theta$$

$$\Rightarrow \sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$

#7

(a) $A = \{z = x+iy \in \mathbb{C} : x \geq 2 \text{ and } y \leq 4\}$

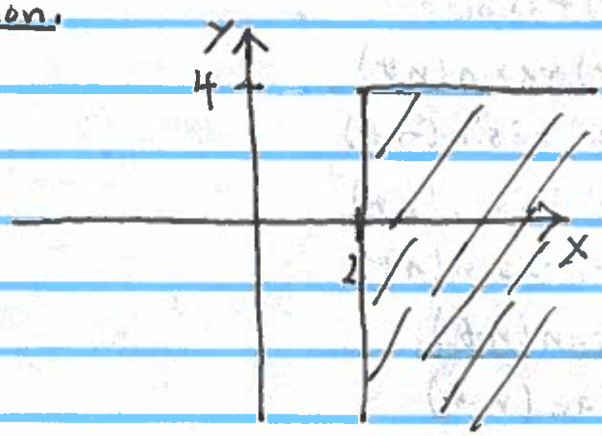
(b) $B = \{z \in \mathbb{C} : |z| < 1 \text{ or } |z-3| \leq 1\}$.

(c) $C = \{x+iy \in \mathbb{C} : x^2 < y\}$

(d) $D = \{z \in \mathbb{C} : \mathrm{Re}(z^2) = 4\}$.

Solution:

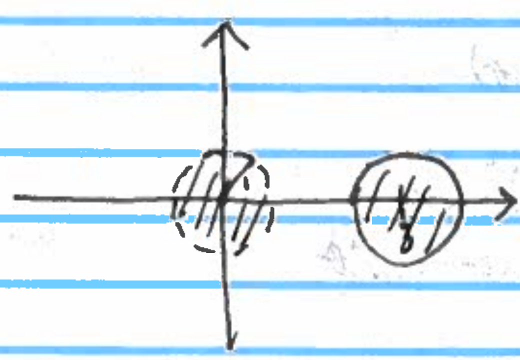
(a)



The interior is shaded, and the boundary is $\{x=2, y\leq 4\} \cup \{x \geq 2, y=4$

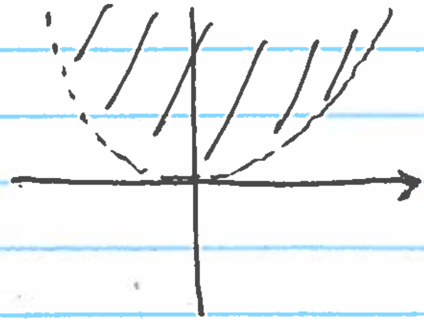
The set is closed, the interior is connected, and the set is unbounded.

(b)



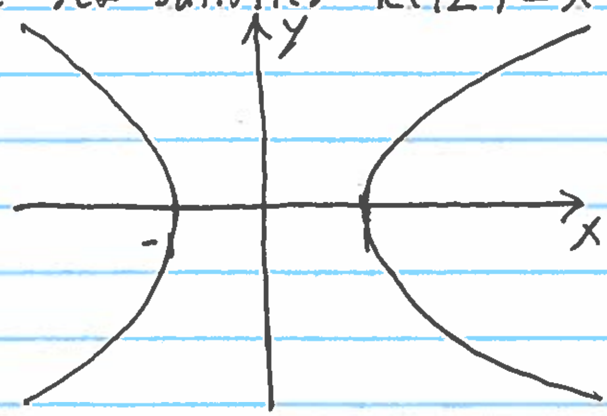
The interior is shaded and the boundary is $\{|z|=1\} \cup \{|z-3|=1\}$.

The set is neither open nor closed. The interior is not connected and the set is bounded.

(c) 

The interior is shaded. The boundary is $\{y=x^2\}$. The set is open, connected and unbounded.

(d) The set satisfies $Re(z^2)=x^2-y^2=1$ which are hyperbolas



The interior is the set $y=x^2$ as is its boundary. The set is closed and is connected based on the definition in the classwork. The set is unbounded.

■

Graduate Problems

#1:

Prove the following

(a) The boundary of any set $D \subseteq \mathbb{C}$ is closed.

(b) For $D \subseteq \mathbb{C}$, show that if $p \in D$, then p is either an interior point of D or a boundary point of D.

(c) Show $D \subseteq \mathbb{C}$ coincides with its boundary if and only if D is closed and has no interior points.

(d) Show that if $D \subseteq \mathbb{C}$ and $E \subseteq \mathbb{C}$ is closed and contains D

then E must contain the boundary of D.

(e) Show that if $D \subseteq \mathbb{C}$ and $S \subseteq \mathbb{C}$ is an open set that is a subset of D, then S must be composed entirely of interior points of D.

Solution:

(a) Let ∂D be the boundary of any set. Therefore,

$$\partial D = \{z \in \mathbb{C} : \text{for all } \varepsilon > 0, \text{ there exists } z_1, z_2 \in \mathbb{C} \text{ such that } z_1 \in N_\varepsilon(z) \cap D \text{ and } z_2 \in N_\varepsilon(z) \cap D^c\}.$$

Consequently,

$$\partial D^c = \{z \in \mathbb{C} : \text{there exists } \varepsilon > 0 \text{ such that for all } z \in N_\varepsilon(z),\ z \in D \text{ or } z \in D^c\}.$$

Therefore, by definition each $z \in \partial D^c$ contains a neighborhood around z and thus ∂D^c is open proving that ∂D is closed.

(b) Suppose $p \in D$. Therefore, for all $\varepsilon > 0$, $p \in N_\varepsilon(p)$. Now, suppose p is not an interior point. Then there exists ε^* such that $N_{\varepsilon^*}(p) \cap D^c \neq \phi$ and $N_{\varepsilon^*}(p) \cap D \neq \phi$ and thus p is a point on the boundary.

(c) Suppose $D = \partial \Omega$. By construction $\partial \Omega$ has no interior points. Now, suppose D has no interior points. It follows from (b) that all points must be interior points.