MTH 317/617
Homework #3

Due Date: September 22, 2023

1 Problems for Everyone

1. Write each of the following functions in the form $w = u(x, y) + iv(x, y)$ and for each function find the domain of definition.

   (a) $f(z) = 3z^2 + 5z + i + 1$

   (b) $f(z) = 1/z$

   (c) $f(z) = \dfrac{z+i}{z^2+1}$

   (d) $f(z) = e^{3z}$

   (e) $f(z) = \dfrac{2z^2+3}{|z-3|}$

   (f) $f(z) = e^z + e^{-z}$

2. For the complex function $f(z) = e^z$:

   (a) Describe the domain of definition and the range.

   (b) Show that $f(-z) = -1/f(z)$.

   (c) Describe the image of the vertical line $\text{Re}(z) = 1$.

   (d) Describe the image of the horizontal line $\text{Im}(z) = \pi/4$.

   (e) Describe the image of the infinite strip $0 \leq \text{Im}(z) \leq \pi/4$.

3. Let $F(z) = z + i$, $G(z) = iz$, and $H(z) = 2z$. Sketch the image of the semi-circle:

$$S = \{z \in \mathbb{C} : |z| = 1, \text{Im}(z) > 0 \text{ and } \text{Re}(z) > 0\}$$

   under the following mappings:

   (a) $F(z)$

   (b) $G(z)$

   (c) $H(z)$

   (d) $G(F(z))$

   (e) $G(H(z))$

   (f) $H(F(z))$

   (g) $F(G(H(z)))$

4. Prove the sequence of complex numbers $z_n = x_n + iy_n$ converges to $z_0 = x_0 + iy_0$ if and only if x_n converges to x_0 and y_n converges to y_0.

5. Prove that the sequence of complex numbers $z_n \to z_0$ if and only if $\overline{z_n} \to \overline{z_0}$.

6. Prove that $z_n \to 0$ if and only if $|z_n| \to 0$.

7. Compute the following limits justifying all steps or prove that the limit does not exist.

   (a) $\lim_{z\to 0} \frac{\operatorname{Im}(z)}{z}$.

   (b) $\lim_{z\to 0} z e^{i\operatorname{Re}(z)}$.

   (c) $\lim_{z\to 0} e^{\frac{1}{z}}$.

   (d) $\lim_{z\to i} \frac{1}{z-i} - \frac{1}{z^2+1}$.

Homework #3

#1

Write each of the following in the form $w = u(x,y) + i\,v(x,y)$ and for each function find the domain of definition.

(c) $f(z) = z+i/z^2+1$

(d) $f(z) = e^z$

(e) $f(z) = (2z^2+3)/|z-3|$

(f) $f(z) = e^z + e^{-z}$.

Solution:

(c) The domain of definition is $\{z \in \mathbb{C} : z \neq \pm i\}$.

$$f(z) = \frac{z+i}{z^2+1} = \frac{1}{z-i} = \frac{1}{x+i(y-1)} = \frac{x-i(y-1)}{x^2+(y-1)^2}.$$

Therefore,

$$u(x,y) = \frac{x}{x^2+(y-1)^2} \text{ and } v(x,y) = -\frac{y-1}{x^2+(y-1)^2}$$

(d) The domain of definition is $\mathbb{C}$ and

$$f(z) = e^z = e^x(\cos(y) + i\sin(y)).$$

Therefore,

$$u(x,y) = e^x\cos(y) \text{ and } v(x,y) = e^x\sin(y).$$

(e) The domain of definition is $\{z \in \mathbb{C} : z \neq 3\}$.

$$f(z) = \frac{2(x+iy)^2+3}{|x+iy-3|}$$

$$= \frac{2x^2+4ixy-2y^2+3}{\sqrt{(x-3)^2+y^2}}$$

Therefore,

$$u(x,y) = \frac{2(x^2-y^2)+3}{\sqrt{(x-3)^2+y^2}} \quad \text{and} \quad v(x,y) = \frac{4xy}{\sqrt{(x-3)^2+y^2}}.$$

(f) The domain of definition is $\mathbb{C}$.

$$f(z) = e^z + e^{-z}$$

$$= e^{x+iy} + e^{-x-iy}$$

$$= e^x(\cos(y)+i\sin(y)) + e^{-x}(\cos(y)-i\sin(y))$$

$$= (e^x+e^{-x})\cos(y) + i(e^x-e^{-x})\sin(y)$$

$$= 2\cosh(x)\cos(y) + 2i\sinh(x)\sin(y)$$

Therefore,

$$u(x,y) = 2\cosh(x)\cos(y) \quad \text{and} \quad v(x,y) = 2\sinh(x)\sin(y).$$

∎

#2.

For the complex function $f(z)=e^z$.

(a) Describe the domain of definition and range.

(c) Describe the image of the vertical line $\mathrm{Re}(z)=1$.

(d) Describe the image of the horizontal line $\mathrm{Im}(z)=\pi/4$.

(e) Describe the image of the infinite strip $0\le \mathrm{Im}(z)\le \pi/4$.

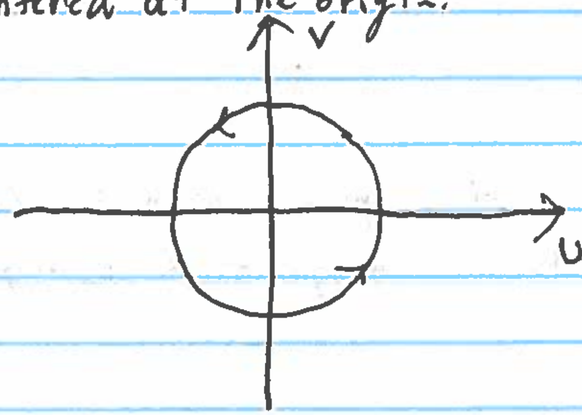
Solution:

(a) Since $e^z=e^x(\cos(y)+i\sin(y))$ it follows that the domain of definition is $\mathbb{C}$ and the range is $\{w\in\mathbb{C} : w\neq 0\}$.

(c). The line $\mathrm{Re}(z)=1$ corresponds to the line $x=1$. Therefore,

$$e^z=e^{x+iy}$$
$$=e(\cos(y)+i\sin(y)),$$

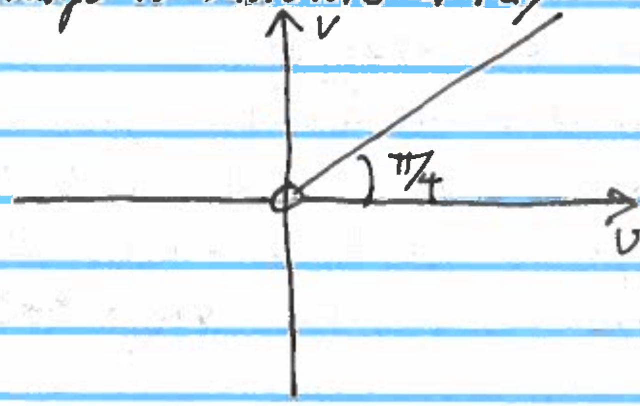
The image is therefore a circle of radius e centered at the origin.



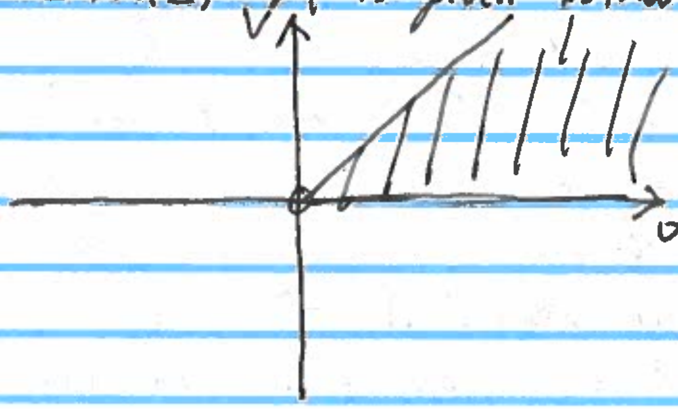
(d) The line $\mathrm{Im}(z)=\pi/4$ corresponds to the line $y=\pi/4$. Therefore,

$$e^z=e^x(\cos(\pi/4)+i\sin(\pi/4))$$
$$=\frac{e^x}{\sqrt{2}}(1+i).$$

The image is therefore a ray



(e). If $y=0$ we have that $e^z = e^x$ and thus the image of $0 \le \mathrm{Im}(z) \le \pi/4$ is given below.



#4

Prove that the sequence of complex numbers $z_n = x_n + iy_n$ converges to $x_0 + iy_0$ if and only if x_n converges to x_0 and y_n converges to y_0.

Solution:

($\Rightarrow$) If $z_n \to z_0$. Then, since

$$|x_n - x_0| \le \sqrt{(x_n-x_0)^2 + (y_n-y_0)^2} = |z_n - z_0|$$

$$|y_n - y_0| \le \sqrt{(x_n-x_0)^2 + (y_n-y_0)^2} = |z_n - z_0|$$

it follows from the squeeze theorem that $|x_n - x_0|, |y_n - y_0| \to 0$

and thus $x_n \to x_0$ and $y_n \to y_0$

($\Leftarrow$) If $x_n \to x_0$ and $y_n \to y_0$ then $|x_n - x_0|, |y_n - y_0| \to 0$ and therefore

$$|z_n - z_0| = \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} \to 0$$

and thus

$$\lim_{n \to \infty} z_n = z_0.$$

$\blacksquare$

#5.

Prove that the sequence of complex numbers $z_n \to z_0$ if and only if $\overline{z_n} \to \overline{z_0}$.

Solution:

Since $z_n \to z_0$ if and only if $|z_n - z_0| \to 0$ and $|z_n - z_0| = |\overline{z_n - z_0}|$ it follows that $z_n \to z_0$ if and only if $|\overline{z_n} - \overline{z_0}| \to 0$. Therefore, $z_n \to z_0$ if and only if $\overline{z_n} \to \overline{z_0}$.

$\blacksquare$

#7.

Compute the following limits justifying all steps or prove that the limit does not exist.

(b) $\lim_{z \to 0} z e^{i Re(z)}$.

(c) $\lim_{z \to 0} e^{1/z}$.

Solution:

(b). Let $z_n \in \mathbb{C}$ satisfy $z_n \to 0$. Therefore,

$$|z_n e^{i Re(z_n)}| = |z_n|$$

$|z_n e^{i Re(z_n)}| \to 0$

Therefore,

$$\lim_{z\to 0} ze^{i\mathrm{Re}(z)} = 0.$$

(c) Let $z_n = i/n$. Therefore, $e^{1/z_n} = e^{-i/n}$, and thus $|e^{1/z_n}| = 1$. Now, if $z_n' = 1/n$ we have that $e^{1/z_n'} = e^n$ which is unbounded. Consequently, $\lim_{z\to 0} e^{1/z}$ does not exist.

■