MTH 317/617
Homework #5

Due Date: October 20, 2023

1 Problems for Everyone

1. Write the following polynomials in the Taylor form, centered at $z = 2$.
   (a) $p(z) = z^5 + 3z + 4$
   (b) $p(z) = z^{10}$
   (c) $p(z) = (z-1)(z-2)^3$.

2. If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ $(a_n \neq 0)$, then its reverse polynomial $p^*(z)$ is given by
$$p^*(z) = \overline{a_n} + \overline{a_{n-1}}z + \ldots + \overline{a_0}z^n.$$
   (a) Show that $p^*(z) = z^n\overline{p(1/\overline{z})}$.
   (b) Show that if $p(z)$ has a zero at $z_0 \neq 0$ then $p^*(z)$ has a zero at $1/\overline{z_0}$.
   (c) Show that for $|z| = 1$, we have $|p(z)| = |p^*(z)|$.

3. Let $f(z)$ be the rational function defined by
$$f(z) = \frac{2z+i}{(z^2+z)(1-z)^2}.$$
   (a) Find all of the poles of this function and their multiplicities.
   (b) Find a partial fraction decomposition of this function.
   (c) If ζ is a pole of $f(z)$ then the coefficient of $\frac{1}{z-\zeta}$ in the partial fraction decomposition is called the residue of $f(z)$ at ζ and is denoted by $\mathrm{Res}(\zeta)$. Find the residues for all of the poles of this function.

4. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be the complex cosine function $f(z) = \cos(z)$.
   (a) Use the Cauchy-Riemann equations to show that $\cos(z)$ is an analytic function and prove that
$$\frac{df}{dz} = -\sin(z).$$
   (b) Compute the real and imaginary parts of the function $f(z^2)$.
   (c) Show that for $z \in \mathbb{C}, \cosh(z) = \cos(iz)$.

5. Prove that if $y \geq 0$ and $x \in \mathbb{R}$ then
$$|\cos(x+iy)| \leq e^y \text{ and } |\sin(x+iy)| \leq e^y.$$

6. Prove the following identities for $z \in \mathbb{C}$:

(a) $\cosh^2(z) - \sinh^2(z) = 1$

(b) $\cosh(z) = \cos(iz)$

(c) $\sinh(z) = -i\sin(iz)$

(d) $|\cosh(z)|^2 = \sinh^2(x) + \cos^2(y)$

(e) $|\sinh(z)|^2 = \sinh^2(x) + \sin^2(y)$

(f) $\overline{\sin(z)} = \sin(\overline{z})$

7. Show that if ξ is any value of

$$-i\log(iz + \sqrt{1 - z^2})$$

then $\sin(\xi) = z$. Likewise, show that if ζ is any value of

$$\frac{i}{2}\log\left(\frac{1 - iw}{1 + iw}\right)$$

then $\tan(\zeta) = w$.

8. **Logarithms**

(a) Write $\log(1 - i)$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

(b) Write $\mathrm{Log}(\sqrt{3} + i)$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

(c) Determine the domain of analyticity for $f(z) = \mathrm{Log}(4 + i - z)$.

(d) Find all solutions $z \in \mathbb{C}$ to the equation $e^{2z} + e^z + 1 = 0$.

9. Find all the values of the given complex power

(a) $(-1)^{3i}$

(b) $3^{2i/\pi}$

(c) $(1 + i)^{1-i}$

(d) $(1 + \sqrt{3}i)^i$

(e) $(-i)^i$

(f) $(ei)^{\sqrt{2}}$

2 Graduate Problems

1. Prove first that

$$1 + e^{i\theta} + e^{2i\theta} + \ldots + e^{in\theta} = \frac{i}{2}\frac{\left(1 - e^{i(n+1)\theta}\right)e^{-i\theta/2}}{\sin(\theta/2)}.$$

Use this result to prove that

$$\frac{1}{2} + \cos(\theta) + \cos(2\theta) + \ldots + \cos(n\theta) = \frac{\sin\left((n + 1/2)\theta\right)}{2\sin(\theta/2)}$$

and

$$\sin(\theta) + \sin(2\theta) + \ldots + \sin(n\theta) = \frac{\cos(\theta/2) - \cos((n + 1/2)\theta)}{2\sin(\theta/2)}.$$

Homework #5

#2

If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, then its reverse polynomial $p^*(z)$ is given by

$$p^*(z) = \overline{a_n} + \overline{a_{n-1}}\, z + \dots + \overline{a_0}\, z^n.$$

(a) Show that $p^*(z) = z^n \overline{p(1/\bar{z})}$.

(b) Show that if $p(z)$ has a zero at $z_0 \neq 0$ then $p^*(z)$ has a zero at $1/\bar{z}_0$.

(c) Show that for $|z| = 1$, we have $|p(z)| = |p^*(z)|$.

Solution:

(a) Computing we have that

$$
\begin{aligned}
z^n \overline{p(1/\bar{z})} &= z^n \left(\bar{a}_0 + \frac{\bar{a}_1}{z} + \dots + \frac{\bar{a}_n}{z^n} \right) \\
&= \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \dots + \bar{a}_n \\
&= \bar{a}_n + \bar{a}_{n-1} z + \dots + \bar{a}_0 z^n. \\
&= p^*(z).
\end{aligned}
$$

(b) Let $z_0 \neq 0$ satisfy $p(z_0) = 0$. Therefore,

$$p^*(1/\bar{z}_0) = \bar{z}_0^{\,n}\, \overline{p(z_0)} = 0$$

(c) Let $z \in \mathbb{C}$ satisfy $|z| = 1$. Therefore, there exists $\theta \in [-\pi, \pi)$ such that $z = e^{i\theta}$. Consequently

$$
\begin{aligned}
|p(z)| &= |p(e^{i\theta})| \\
&= |e^{ni\theta}\, \overline{p(e^{i\theta})}| \\
&= |e^{ni\theta}\, \overline{p(1/\overline{e^{i\theta}})}| \\
&= |p^*(z)|
\end{aligned}
$$

$\blacksquare$

#4.

Let $f: \mathbb{C} \to \mathbb{C}$ be the complex cosine function $f(z) = \cos(z)$.

(a) Use the Cauchy-Riemann equations to show that $\cos(z)$ is an analytic function and prove that

$$\frac{df}{dz} = -\sin(z).$$

(b) Compute the real and imaginary parts of the function $f(z^2)$.

(c) Show that for $z \in \mathbb{C}$, $\cosh(z) = \cos(iz)$.

Solution:

(a) $$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{e^{ix-y} + e^{-ix+y}}{2}$$

$$= \frac{e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x))}{2}$$

$$= \cosh(y)\cos(x) + i\sinh(y)\sin(x)$$

$$= u(x,y) + iv(x,y)$$

Therefore,

$$\frac{\partial u}{\partial x} = -\cosh(y)\sin(x), \quad \frac{\partial v}{\partial x} = -\sinh(y)\cos(x)$$

$$\frac{\partial u}{\partial y} = \sinh(y)\cos(x), \quad \frac{\partial v}{\partial y} = -\cosh(y)\sin(x)$$

and thus the Cauchy-Riemann equations are satisfied. Moreover

$$f'(z) = -\cosh(y)\sin(x) - i\sinh(y)\cos(x)$$

$$= -\sin(z).$$

(b) Since $z^2 = x^2 - y^2 + 2ixy$ it follows that

$$\cos(z^2) = \cosh(2xy)\cos(x^2-y^2) + i\sinh(2xy)\sin(x^2-y^2).$$

(c)
$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$
$$= \frac{e^{-i\cdot i\cdot z} + e^{i\cdot i\cdot z}}{2}$$
$$= \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$
$$= \cos(iz)$$

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#5

Prove that if $y \geq 0$ and $x \in \mathbb{R}$ then
$$|\cos(x+iy)| \leq e^y \text{ and } |\sin(x+iy)| \leq e^y$$

Solution:

$$|\cos(x+iy)| = \frac{|e^{ix-y} + e^{-ix+y}|}{2}$$
$$\leq \frac{|e^{ix}e^{-y}| + |e^{-ix}e^{y}|}{2}$$
$$\leq \frac{e^{-y}+e^y}{2}$$
$$\leq e^y$$

$$|\sin(x+iy)| = |e^{ix-y} - e^{-ix+y}|/|2i|$$
$$\leq |e^{ix}|e^{-y} + |e^{-ix}|e^{y}/2$$
$$\leq \frac{e^{-y}+e^y}{2}$$
$$\leq e^y$$

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#6

Prove the following identities for $z \in \mathbb{C}$.

(a) $\cosh^2(z) - \sinh^2(z) = 1$

(c) $\sinh(z) = -i\sin(iz)$

(d) $|\cosh(z)|^2 = \sinh^2(x) + \cos^2(y)$

(f) $\overline{\sin(z)} = \sin(\bar{z})$.

Solution:

(a) $$\cosh^2(z) - \sinh^2(z) = \frac{(e^{z} + e^{-z})^2}{4} - \frac{(e^{z} - e^{-z})^2}{4}$$
$$= \frac{e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}}{4}$$
$$= 1$$

(c) $$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$
$$= \frac{e^{-iiz} - e^{iiz}}{2i} \cdot i$$
$$= -i\frac{(e^{i(iz)} - e^{-i(iz)})}{2i}$$
$$= -i\sin(iz).$$

(f) $$\overline{\sin(z)} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)}$$
$$= \frac{e^{-i\bar{z}} - e^{i\bar{z}}}{-2i}$$
$$= \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i}$$
$$= \sin(\bar{z})$$

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#2

Show that if ξ is any value of

$$\frac{i}{2}\log\left(\frac{1-iw}{1+iw}\right)$$

then $\tan(\xi)=w$.

Solution:

$$\tan(\xi)=\tan\left(\frac{i}{2}\log\left(\frac{1-iw}{1+iw}\right)\right)$$

Now,

$$e^{i\xi}=e^{-\frac{1}{2}\log\left(\frac{1-iw}{1+iw}\right)}=\left(\frac{1+iw}{1-iw}\right)^{1/2}$$

$$e^{-i\xi}=e^{\frac{1}{2}\log\left(\frac{1-iw}{1+iw}\right)}=\left(\frac{1-iw}{1+iw}\right)^{1/2}.$$

Therefore,

$$\tan(\xi)=\frac{e^{i\xi}-e^{-i\xi}}{(e^{i\xi}+e^{-i\xi})i}$$

$$=\frac{\left(\frac{1+iw}{1-iw}\right)^{1/2}-\left(\frac{1-iw}{1+iw}\right)^{1/2}}{\left(\left(\frac{1+iw}{1-iw}\right)^{1/2}+\left(\frac{1-iw}{1+iw}\right)^{1/2}\right)i}$$

$$=\frac{(1+iw)-(1-iw)}{(1+iw+1-iw)i}$$

$$=\frac{2iw}{2i}$$

$$=w. \qquad \blacksquare$$

Graduate Problem

Prove that

$$1+e^{i\theta}+e^{2i\theta}+\dots+e^{in\theta}=\frac{i}{2}\,\frac{\left(1-e^{i(n+1)\theta}\right)e^{-i\theta/2}}{\sin(\theta/2)}$$

Use this result to prove that

$$\frac{1}{2}+\cos\theta+\cos(2\theta)+\dots+\cos(n\theta)=\frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}$$

and

$$\sin\theta+\sin(2\theta)+\dots+\sin(n\theta)=\frac{\cos(\theta/2)-\cos((n+1/2)\theta)}{2\sin(\theta/2)}$$

Solution:

Recognizing a geometric series we have that

$$\begin{aligned}1+e^{i\theta}+e^{2i\theta}+\dots+e^{in\theta}&=\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\\&=\frac{1-e^{i(n+1)\theta}}{\left(e^{-i\theta/2}-e^{i\theta/2}\right)e^{i\theta/2}}\\&=\frac{\left(1-e^{i(n+1)\theta}\right)e^{-i\theta/2}}{-2i\sin(\theta/2)}\\&=\frac{i}{2}\,\frac{\left(1-e^{i(n+1)\theta}\right)e^{-i\theta/2}}{\sin(\theta/2)}\end{aligned}$$

Furthermore,

$$\begin{aligned}\left(1-e^{i(n+1)\theta}\right)e^{-i\theta/2}&=e^{-i\theta/2}-e^{i(n+\frac{1}{2})\theta}\\&=\cos(\theta/2)-i\sin(\theta/2)-\cos((n+\tfrac{1}{2})\theta)-i\sin((n+\tfrac{1}{2})\theta)\end{aligned}$$

$$\Rightarrow \frac{i}{2}\,\frac{\left(1-e^{i(n+1)\theta}\right)e^{-i\theta/2}}{\sin(\theta/2)}=\frac{i}{2\sin(\theta/2)}\left(\cos\theta/2-\cos((n+\tfrac{1}{2})\theta)\right)+\frac{1}{2\sin(\theta/2)}\bigl(\sin(\theta/2)+\sin((n+\tfrac{1}{2})$$

Therefore,

$$1+\cos\theta+\cos(2\theta)+\dots+\cos(n\theta)=\frac{1}{2}+\frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\theta/2)}$$

$$\Rightarrow \tfrac{1}{2}+\cos\theta+\cos(2\theta)+\dots+\cos(n\theta)=\frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\theta/2)}.$$

Moreover,

$$\sin\theta+\sin 2\theta+\dots+\sin(n\theta)=\frac{\cos(\theta/2)-\cos((n+\frac{1}{2})\theta)}{2\sin(\theta/2)}$$

$\blacksquare$