MTH 317/617
Homework #8

Due Date: November 20, 2023

1 Problems for Everyone

1. For the following functions find the first four terms of the Taylor series about z_0 and determine the radius of convergence of the series

   (a) $\dfrac{1}{1+z}$, $z_0 = 0$.
   (b) e^{-z^2}, $z_0 = 0$.
   (c) $z^3 \sin(3z)$, $z_0 = 0$.
   (d) $z^3 \sin(3z)$, $z_0 = 0$.
   (e) $\dfrac{1+z}{1-z}$, $z_0 = i$.
   (f) $\dfrac{e^z}{3-2z}$, $z_0 = 0$.
   (g) $\dfrac{z}{(1-z)^2}$, $z_0 = 0$.

2. pg. 212, #3

3. pg. 212, #4

4. pg. 212, #6, follow the problem's hint and use Taylor's theorem to expand the numerator for each integral.

5. Find the first four terms of the Laurent series for the function $f(z) = \dfrac{1}{z+z^2}$ in each of the following domains

   (a) $0 < |z| < 1$
   (b) $|z| > 1$
   (c) $0 < |z+1| < 1$
   (d) $1 < |z+1|$

6. Find the first four terms of the Laurent series for the following functions about the indicated point

(a) $\frac{e^z - 1}{z^2}$; $z_0 = 0$

(b) $\frac{z^2}{z^2 - 1}$; $z_0 = 1$

(c) $\frac{\sin(z)}{(z - \pi)^2}$; $z_0 = \pi$

(d) $\frac{z}{(\sin(z))^2}$; $z_0 = 0$

(e) $\frac{1}{e^z - 1}$; $z_0 = 0$

7. Evaluate the following contour integrals:

(a) $\int_{|z|=1} \frac{z^2 + 3z - 1}{z(z^2 - 3)} dz$

(b) $\int_{|z|=1} \frac{\sin(z)}{z^6} dz$

(c) $\int_{|z|=4} z \tan(z) dz$

(d) $\int_{|z|=1} \frac{e^{z^2}}{z^6} dz$

(e) $\int_{|z|=1} z^4 (e^{z^{-1}} + z^2) dz$

(f) $\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{z^{-1}} dz$

(g) $\int_{|z|=1} \frac{1}{z^2}(e^z - 1) dz$

Homework #8

#1

For the following functions find the first four terms of the Taylor series about z_0 and determine the radius of convergence of the series.

(b) e^{-z^2}, $z_0=0$

(c) $z^3\sin(3z)$, $z_0=0$

(e) $\frac{1+z}{1-z}$, $z_0=i$

(f) $\frac{e^z}{3-2z}$, $z_0=0$

Solution:

(b) The radius of convergence is $R=\infty$. Expanding we have that

$$e^{-z^2}=1-z^2+\frac{1}{2}z^4-\frac{1}{6}z^6+\dots$$

(c) The radius of convergence is $R=\infty$ Expanding we have that

$$z^3\sin(3z)=z^3\left(3z-\frac{1}{6}(3z)^3+\frac{1}{5!}(3z)^5-\frac{1}{7!}(3z)^7+\dots\right)$$
$$=3z^4-\frac{1}{6}3^3z^6+\frac{1}{5!}3^5z^8-\frac{1}{7!}3^7z^{10}+\dots$$

(e) The singularity is located at $z=1$. Thus the radius of convergence is the distance between $z=1$ and $z=i$ which is $R=\sqrt{2}$. Letting $w=z-i$ we have that $w+i=z$. Therefore, for $|w|<\sqrt{2}$ we have

$$\frac{1+z}{1-z}=\frac{1+i+w}{1-i-w}=\frac{1+i+w}{(1-i)\left(1-\frac{w}{1-i}\right)}$$

$$\Rightarrow \frac{1+z}{1-z} = \frac{1+i+w}{(1-i)}\left(1+\frac{w}{1-i}+\frac{w^2}{(1-i)^2}+\frac{w^3}{(1-i)^3}+\cdots\right)$$

$$= \frac{1+i}{1-i} + \left(\frac{1}{1-i}+\frac{1+i}{(1-i)^2}\right)w + \left(\frac{1+i}{(1-i)^3}+\frac{1}{(1-i)^2}\right)w^2 + \left(\frac{1+i}{(1-i)^4}\frac{1}{(1-i)^3}\right)w^3+\cdots$$

$$= \frac{1+i}{1-i} + \frac{2}{(1-i)^2}w + \frac{2}{(1-i)^3}w^2 + \frac{2}{(1-i)^4}w^3+\cdots$$

$$= \frac{1+i}{1-i} + \frac{2}{(1-i)^2}(z-i) + \frac{2}{(1-i)^3}(z-i)^2 + \frac{2}{(1-i)^4}(z-i)^3+\cdots$$

(f) This function has a singularity at $z = 3/2$ and thus the radius of convergence is $3/2$. Expanding, we have that

$$\frac{e^z}{3-2z} = \frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots}{3(1-2z/3)}$$

$$= \frac{[1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots]}{3}\left(1+2z/3+4z^2/9+8z^3/27+\cdots\right)$$

$$= \frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}+1\right)z+\frac{1}{3}\left(\frac{1}{2}+\frac{4}{9}+\frac{2}{3}\right)z^2$$

$$+\frac{1}{3}\left(\frac{8}{27}+\frac{4}{9}+\frac{1}{3}+\frac{1}{6}\right)z^3+\cdots$$

$\blacksquare$

#4

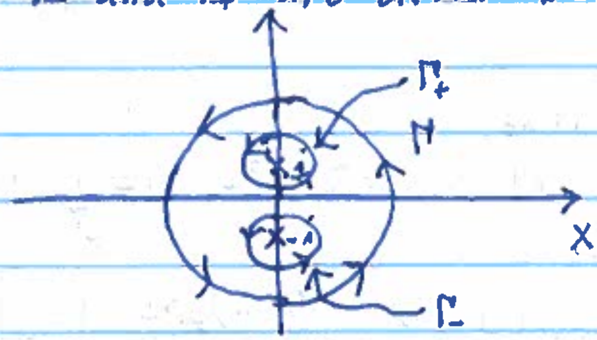
Evaluate $\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz$, where Γ is the circle $|z|=3$ transversed once clockwise.

Solution:

Computing, we have that

$$\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz = \int_{\Gamma_-} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz + \int_{\Gamma_+} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz,$$

where Γ_- and Γ_+ are drawn below:



Taylor expanding about $z=i$ we have:

$$\frac{e^{iz}}{(z+i)^2} = \frac{e^{-1}}{(2i)^2} + \left.\frac{(z+i)^2 ie^{iz} - e^{iz}\cdot 2(z+i)}{(z+i)^4}\right|_{z=i}(z-i)+\dots$$

$$= \frac{e^{-1}}{-4} + \frac{(-4)ie^{-1} - e^{-1}\cdot 4i}{-16}(z-i)+\dots$$

$$= \frac{e^{-1}}{-4} + \frac{e^{-1}i}{2}(z-i)+\dots$$

$$\Rightarrow \int \frac{e^{iz}}{(z-i)^2}\,dz = 2\pi i \cdot {}^{i}\!/_{2} = -\pi e^{-1}$$

Taylor expanding about $z=-i$ we have:

$$\frac{e^{iz}}{(z-i)^2} = \frac{e^{1}}{-4} + \left.\frac{(z-i)^2 ie^{iz} - e^{iz}2(z-i)}{(z-i)^4}\right|_{z=-i}(z+i)+\dots$$

$$= -\frac{e^{1}}{4} + \frac{\cancelto{0}{(-4)ie^{1} + e^{1}4}}{(-16)}(z-i)+\dots$$

$$\Rightarrow \int_{\Gamma_-} \frac{e^{iz}}{(z^2+1)^2}\,dz = 0$$

$$\Rightarrow \int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz = -\pi/e. \quad \blacksquare$$

#5

Find the first four terms of the Laurent series for the function $f(z) = 1/z+z^2$ in each of the following domains.

(a) $0<|z|<1$

(b) $|z|>1$

(c) $0<|z+1|<1$

(d) $1<|z+1|$.

Solution:

(a) Expanding we have that

$$\frac{1}{z+z^2} = \frac{1}{z(1+z)} = \frac{1}{z}(1-z+z^2-z^3+\dots) = \frac{1}{z} - 1 + z - z^2 + \dots$$

(b) $$\frac{1}{z+z^2} = \frac{1}{z^2(1+1/z)} = \frac{1}{z^2}(1 - 1/z + 1/z^2 - 1/z^3 + \dots) = \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \frac{1}{z^5} + \dots$$

(c) Letting $w = z+1$ we have that $z = w-1$ and

$$\frac{1}{z(1+z)} = \frac{1}{w(w-1)} = \frac{-1}{w}(1+w+w^2+w^3+\dots) = \frac{-1}{w} - 1 - w - w^2 + \dots$$

$$\Rightarrow \frac{1}{z(1+z)} = \frac{-1}{z+1} - 1 - (z+1) - (z+1)^2 + \dots$$

(d) $$\frac{1}{z(1+z)} = \frac{1}{w^2(1-1/w)} = \frac{1}{w^2}(1 + 1/w + 1/w^2 + 1/w^3 + \dots) = \frac{1}{w^2} + \frac{1}{w^3} + \frac{1}{w^4} + \frac{1}{w^5} + \dots$$

$$\Rightarrow \frac{1}{z(1+z)} = \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} + \frac{1}{(z+1)^5} + \dots$$

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#6

Find the first four terms of the Laurent series for the following functions about the indicated point.

(a) $\frac{e^{z-1}}{z^2}$, $z_0 = 0$

(b) z^2/z^2-1, $z_0 = 1$

(c) $\sin(z)/(z-\pi)^2$, $z_0 = \pi$

(d) $z/\sin^2(z)$, $z_0 = 0$

Solution:

(a) $$\frac{e^{z-1}}{z^2} = e^{-1}\frac{(1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\dots)}{z^2}$$

$$= \frac{e^{-1}}{z^2} + \frac{e^{-1}}{z} + \frac{e^{-1}}{2} + e^{-1}z + \dots$$

(b) Let $w = z-1 \Rightarrow z = 1+w$. We therefore have

$$\frac{z^2}{z^2-1} = \frac{(1+w)^2}{w(2+w)} = \frac{1+2w+w^2}{2w(1+w/2)} = \frac{1+2w+w^2}{2w}(1-w/2+w^2/4-w^3/8+\dots)$$

$$\Rightarrow \frac{z^2}{z^2-1} = \frac{1}{2w} + \left(1-\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{2}+\frac{1}{8}\right)w + \left(\frac{-1}{16}+\frac{1}{4}-\frac{1}{4}\right)w^2 + \dots$$

$$= \frac{1}{2w} + \frac{3}{4} + \frac{1}{8}w - \frac{1}{16}w^2 + \dots$$

$$= \frac{1}{2(z-1)} + \frac{3}{4} + \frac{1}{8}(z-1) - \frac{1}{16}(z-1)^2 + \dots$$

(c) Letting $w = z - \pi$ we have that

$$\frac{\sin(z)}{(z-\pi)^2} = \frac{-\sin(w)}{w^2} = \frac{-w + w^3/6 - w^5/5! + w^7/7! + \ldots}{w^2} = \frac{-1}{w} + \frac{w}{6} - \frac{w^3}{5!} + \frac{w^5}{7!} + \ldots$$

$$\Rightarrow \frac{\sin(z)}{(z-\pi)^2} = \frac{-1}{(z-\pi)} + \frac{(z-\pi)}{6} - \frac{(z-\pi)^3}{5!} + \frac{(z-\pi)^5}{7!} + \ldots$$

(d) $$\frac{z}{\sin^2(z)} = \frac{z}{(z - z^3/6 + z^5/5! - z^7/7! + \ldots)^2} = \frac{z}{z^2(1-(z^2/6 - z^4/5! + z^6/7! + \ldots))^2}.$$

Letting $g(z) = z^2/6 - z^4/5! + z^6/7! + \ldots$ we have that

$$\frac{1}{(1-g(z))^2} = \frac{d}{dg}\frac{1}{(1-g(z))} = \frac{d}{dg}(1 + g(z) + g(z)^2 + g(z)^3 + \ldots)$$

$$\Rightarrow \frac{1}{(1-g(z))^2} = 1 + 2g(z) + 3g(z)^2 + 4g(z)^3 + \ldots$$

$$= 1 + 2(z^2/6 - z^4/5! + z^6/7! + \ldots) + 3(z^2/6 - z^4/5! + z^6/7! + \ldots)^2$$
$$+ 4(z^2/6 - z^4/5! + z^6/7! + \ldots)^3 + \ldots$$

$$= 1 + \tfrac{1}{3}z^2 + (-2/5! + 3/36)z^4 + (2/7! - 6 \cdot 3/6 \cdot 5! + 4/6^3)z^6 + \ldots$$

Therefore, $z/\sin^2(z)$ is given by

$$\frac{z}{\sin^2(z)} = \frac{1}{z} + \frac{1}{3}z + \frac{1}{15}z^3 + \frac{2}{189}z^5 + \ldots$$

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#7.

Evaluate the following integrals

(a) $\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)} dz$

(b) $\int_{|z|=1} \frac{\sin(z)}{z^6} dz$

(d) $\int_{|z|=1} \frac{e^{z^2}}{z^6} dz$

(f) $\int_{|z|=1} \cos(1/z^2) e^{1/z} dz$

Solution:

(a) $$\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)} dz = 2\pi i \left. \frac{z^2+3z-1}{z^2-3} \right|_{z=0} = \frac{2\pi i}{3}$$

(b) $$\int_{|z|=1} \frac{\sin(z)}{z^6} dz = \int_{|z|=1} \frac{z - z^3/6! + z^5/5! + \ldots}{z^6} dz = \frac{2\pi i}{5!}$$

(d) $$\int_{|z|=1} \frac{e^{z^2}}{z^6} dz = \int_{|z|=1} \frac{1+z^2+\frac{1}{2}z^4+\frac{1}{3!}z^6+\ldots}{z^6} dz = 0.$$

(f) $$\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{z^{-1}} dz = \int_{|z|=1} \left(1 - \frac{1}{2z^4} + \frac{1}{4!z^8} + \ldots\right)\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots\right) dz$$

$$= \int_{|z|=1} \frac{1}{z} dz$$

$$= 2\pi i.$$

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