MTH 317/617
Homework #9

Due Date: December 1, 2023

1 Problems for Everyone

1. Let f be analytic except at an isolated singularity z_0 and suppose that the Laurent series for f about z_0 is given by
$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j.$$
Show that the coefficients a_j are given by
$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{j+1}} dz,$$
where Γ is any closed contour containing z_0.

2. Prove that if f has a simple pole at z_0 then
$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

3. Let $f(z) = P(z)/Q(z)$, where the functions $P(z)$ and $Q(z)$ are both analytic at z_0, and Q has a simple zero at z_0, while $P(z_0) \neq 0$. Prove that
$$\mathrm{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

4. Prove that if f has a pole of order m at z_0, then
$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

5. Suppose that f is analytic and has a zero of order m at the point z_0. Show that the function $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with
$$\mathrm{Res}(g; z_0) = m.$$

6. Determine all of the isolated singularities of each of the following functions and compute the residue at each singularity.

(a) $f(z) = \dfrac{e^{3z}}{z-2}$

(b) $f(z) = \dfrac{z+1}{z^2-3z+2}$

(c) $f(z) = \dfrac{1+e^z}{z^2} + \dfrac{2}{z}$

(d) $f(z) = \dfrac{\sin(z^2)}{z^2(z^2+1)}$

(e) $f(z) = \dfrac{1-\cos(z)}{z^2}$

(f) $f(z) = \dfrac{1}{z\sin(z)}$

(g) $f(z) = \sin\left(\dfrac{1}{3z}\right)$

7. Evaluate the following contour integrals

(a) $\displaystyle\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}dz$

(b) $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z^6}dz$

(c) $\displaystyle\int_{|z|=4} z\tan(z)dz$

(d) $\displaystyle\int_{|z|=1} \frac{e^{z^2}}{z^6}dz$

(e) $\displaystyle\int_{|z|=1} z^4\left(e^{z^{-1}}+z^2\right)dz$

(f) $\displaystyle\int_{|z|=1} \cos\left(\frac{1}{z^2}\right)e^{z^{-1}}dz$

(g) $\displaystyle\int_{|z|=1} \frac{1}{z^2(e^z-1)}dz$

8. Verify each of the following integrals by writing the integral as a contour integral in the complex plane:

(a) $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin(\theta)}d\theta = \frac{2\pi}{\sqrt{3}}$

(b) $\displaystyle\int_0^{2\pi} \frac{8}{5+2\cos(\theta)}d\theta = \frac{16\pi}{\sqrt{21}}$

(c) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1+\sin^2(\theta)}d\theta = \sqrt{2}\pi$

Homework #9

#3

Let $f(z) = P(z)/Q(z)$, where the functions $P(z)$ and $Q(z)$ are analytic at z_0 and Q has a simple zero at z_0, while $P(z_0) \neq 0$. Prove that

$$\operatorname{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}$$

Solution:

Since P and Q are analytic and $Q(z_0) = 0$ it follows that near z_0

$$f(z) = \frac{P(z_0) + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots}{Q'(z_0)(z-z_0) + b_2(z-z_0)^2 + \ldots}$$

$$= \frac{P(z_0) + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots}{(z-z_0)(Q'(z_0) + b_2(z-z_0) + \ldots)}$$

$$\Rightarrow \operatorname{Res}(f; z_0) = \left. \frac{P(z_0) + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots}{Q'(z_0) + b_2(z-z_0) + \ldots} \right|_{z=z_0}$$

$$= \frac{P(z_0)}{Q'(z_0)}.$$

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#4

Prove that if f has a pole of order m at z_0, then

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

~~proof~~:

Expanding, we have that

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$\Rightarrow (z-z_0)^m f(z) = a_{-m} + \dots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \dots$$

Therefore,

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = (m-1)!\, a_{-1} + m\cdot(m-1)\cdots 2\, a_0 (z-z_0) + (m+1)(m)(m-1)\cdots 3\, a_1 (z-z_0)^2 + \dots$$

$$\Rightarrow \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]\Bigg|_{z_0} = a_{-1} = \mathrm{Res}(f; z_0).$$

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#5

Suppose that f is analytic and has a zero of order m at the point z_0. Show that $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with

$$\mathrm{Res}(g; z_0) = m$$

Solution:

Since f has a zero of order m it follows that

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + a_{m+2}(z-z_0)^{m+2} + \dots$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{a_m m(z-z_0)^{m-1} + a_{m+1}(m+1)(z-z_0)^m + \dots}{a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \dots}$$

$$= \frac{(z-z_0)^{m-1}(a_m \cdot m + a_{m+1}(m+1)(z-z_0) + \dots)}{(z-z_0)^m(a_m + a_{m+1}(m+1)(z-z_0) + \dots)}$$

$$= \frac{(a_m \cdot m + a_{m+1}(m+1)(z-z_0) + \dots)}{(z-z_0)(a_m + a_{m+1}(m+1)(z-z_0) + \dots)}$$

Therefore,

$$\operatorname{Res}(g; z_0) = \left.\frac{(a_m \cdot m + a_{m+1}(m+1)(z-z_0) + \dots)}{(a_m + a_{m+1}(m+1)(z-z_0) + \dots)}\right|_{z=z_0}$$

$$= m$$

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#6

Determine all of the isolated singularities of each of the following functions and compute the residue at each singularity.

(c) $f(z) = \frac{1+e^z}{z^2} + \frac{2}{z}$

(d) $f(z) = \frac{\sin(z^2)}{z^2(z^2+1)}$

(e) $f(z) = \frac{1-\cos(z)}{z^2}$

(g) $f(z) = \sin(1/3z)$

Solution:

(c) $\dfrac{1+e^z}{z^2}+\dfrac{2}{z}=\dfrac{1+1+z+z^2+\cdots}{z^2}+\dfrac{2}{z}$

$\Rightarrow \text{Res}\left(\dfrac{1+e^z}{z^2};0\right)=3$

(d) $f(z)=\dfrac{\sin(z^2)}{z^2(z^2+1)}$ has singularities at 0 and $\pm i$.

$z=0$:

$\dfrac{\sin(z^2)}{z^2(1+z^2)}=\dfrac{(z^2-z^6/3!+\ldots)}{z^2}(1-z^2+z^4+\ldots)$

$\Rightarrow \text{Res}(f;0)=0$

$z=i$:

$\dfrac{\sin(z^2)}{z^2(z+i)(z-i)}\Rightarrow \text{Res}(f;i)=\left.\dfrac{\sin(z^2)}{z^2(z+i)}\right|_{z=i}=\dfrac{\sin(-1)}{-2i}$.

$z=-i$:

$\dfrac{\sin(z^2)}{z^2(z+i)(z-i)}\Rightarrow \text{Res}(f;-i)=\left.\dfrac{\sin(z^2)}{z^2(z-i)}\right|_{z=-i}=\dfrac{\sin(-1)}{2i}$

(e) $f(z)=1-\cos(z)/z^2$ has a singularity at $z=0$. Therefore,

$\dfrac{1-\cos(z)}{z^2}=\dfrac{1-1+z^2/2-z^4/4!+\ldots}{z^2}$

$=\dfrac{1}{2}-\dfrac{z^2}{4!}+\ldots$

$\Rightarrow \text{Res}(f;0)=0.$

(g) $f(z) = \sin(1/3z)$ has a singularity at $z=0$. Moreover,

$$\sin(1/3z) = \frac{1}{3z} - \frac{1}{3!(3z)^3} + \frac{1}{5!(3z)^5} + \dots$$

$$\Rightarrow \mathrm{Res}(f;0) = \frac{1}{3}.$$

$\blacksquare$

#7

Evaluating the following contour integrals

(a) $\displaystyle\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}\,dz$

(b) $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z^6}\,dz$

(e) $\displaystyle\int_{|z|=1} z^4(e^{z^{-1}}+z^2)\,dz$

(f) $\displaystyle\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{z^{-1}}\,dz$

Solution:

(a) $$\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}\,dz = 2\pi i \left(\frac{z^2+3z-1}{z^2-3}\right)\Bigg|_{z=0} = \frac{2\pi i}{3}.$$

(b) $$\int_{|z|=1} \frac{\sin(z)}{z^6}\,dz = \int_{|z|=1} \frac{(z - z^3/3! + z^5/5! + \dots)}{z^6}\,dz$$

$$= \int_{|z|=1} \frac{1}{5!z}\,dz$$

$$= \frac{2\pi i}{5!}$$

(e) $\int_{|z|=1} z^4(e^{z^{-1}}+z^2)\,dz = \int_{|z|=1} z^4 e^{z^{-1}}\,dz$

$$= \int_{|z|=1} z^4\left(1+1/z+1/2z^2+1/3!z^3+1/4!z^4+1/5!z^5+\dots\right)dz$$

$$= \int_{|z|=1} \frac{1}{5!z}\,dz$$

$$= \frac{2\pi i}{5!}$$

(f). $\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{z^{-1}} dz = \int_{|z|=1} \left(1-\frac{1}{2z^4}+\dots\right)\left(1+\frac{1}{z}+\frac{1}{2z^2}+\dots\right)dz$

$$= \int_{|z|=1} \frac{1}{z}\,dz$$

$$= 2\pi i$$

#8

Verify each of the following integrals by writing the integral in the complex plane:

(a) $\int_0^{2\pi} \frac{1}{2+\sin(\theta)}\,d\theta = \frac{2\pi}{\sqrt{3}}$

(b) $\int_0^{2\pi} \frac{8}{5+2\cos\theta}\,d\theta = \frac{16\pi}{\sqrt{21}}$

(c) $\int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \sqrt{2}\pi.$

Solution:

(a) Letting $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = -ie^{-i\theta}dz.$

Therefore,

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \int_\Gamma \frac{-i}{z(2+(z-z^{-1})/2i)}\,dz,$$

$$= \int_\Gamma \frac{2}{z(4i+z-z^{-1})}\,dz,$$

where Γ is the unit circle about the origin.

$$\Rightarrow \int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \int_\Gamma \frac{2}{(z^2+4iz-1)}\,dz.$$

The function $f(z) = \frac{2}{z^2+4iz-1}$ has singularities at

$$z = \frac{-4i \pm \sqrt{-16+4}}{2}$$

$$= -2i \pm \sqrt{3}i$$

Therefore,

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \operatorname{Res}\left(\frac{2}{z^2+4iz-1}; -2i+\sqrt{3}i\right) 2\pi i$$

$$= \left.\frac{2}{2z+4i}\right|_{-2i+\sqrt{3}i} 2\pi i$$

$$= \frac{4\pi i}{2\sqrt{3}i}$$

$$= \frac{2\pi}{\sqrt{3}}$$

(b) Letting $z = e^{i\theta}$ we have that $dz = ie^{i\theta} d\theta \Rightarrow d\theta = -iz^{-1} dz$.

Therefore,

$$\int_0^{2\pi} \frac{8}{5+2\cos\theta} d\theta = \int_\Gamma \frac{-i8}{z(5+z+z^{-1})} dz,$$

where Γ is the unit circle about the origin. The integrand has singularities when

$$z^2 + 5z + 1 = 0$$

$$\Rightarrow z = \frac{-5 \pm \sqrt{21}}{2}$$

Therefore,

$$\int_0^{2\pi} \frac{8}{5+2\cos\theta} d\theta = \mathrm{Res}\left(\frac{-i8}{z^2+5z1}; \frac{-5+\sqrt{21}}{2}\right) 2\pi i$$

$$= \left(\frac{-i8}{2z+5}\bigg|_{\frac{-5+\sqrt{21}}{2}}\right) 2\pi i$$

$$= \frac{16\pi}{\sqrt{21}}$$

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(c) Letting $z = e^{i\theta}$ we have that $dz = ie^{i\theta} d\theta \Rightarrow -iz^{-1} dz$. Therefore,

$$\int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta} d\theta = \int_\Gamma \frac{-i}{z(1+\frac{(z-z^{-1})^2}{(2i)^2})} dz$$

$$= \int_\Gamma \frac{4i}{z(-4+z^2-2+1/z^2)} dz$$

$$= \int_\Gamma \frac{4iz^2}{z(z^4-6z^2+1)}$$

$$= \int_\Gamma \frac{4iz}{z^4-6z^2+1} dz$$

$\vdots$

(Not graded)