MTH 317/617
Homework #4

Due Date: October 6, 2023

1 Problems for Everyone

1. Show that each of the following functions is nowhere differentiable using the definition of the derivative, i.e. do not use the Cauchy Riemann equations.
   (a) $f(z) = \mathrm{Re}(z)$
   (b) $f(z) = \mathrm{Im}(z)$
2. Let $P(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. Prove that
$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_n}$$
3. Find real constants a, b, c, d so that the given function is analytic
   (a) $f(z) = 3x - y + 5 + i(ax + by - 3)$.
   (b) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$.
4. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic then
$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2.$$
5. Verify that each given function u is harmonic (in the region where it is defined) and then find the function v such that $f(z) = u(x, y) + iv(x, y)$ is analytic.
   (a) $u = e^x \sin(y)$
   (b) $u = xy - x + y$
   (c) $u = \sin(x)\cosh(y)$
   (d) $u = \mathrm{Im}\left(\exp(z^2)\right)$

2 Graduate Problems

1. A set A is said to be an ordered set provided it contains a subset P with the following two properties
   - For any $x \in A$, either $x \in P$ or $-x \in P$ (but not both).
   - If $x, y \in P$ then both $xy \in P$ and $x + y \in P$.

   In $\mathbb{R}$ the set P is the set of positive numbers. In $\mathbb{R}$ we say $x > y$, if and only if $x - y \in P$.

   Prove that $\mathbb{C}$ is not an ordered set.