MTH 317/617
Homework #5

Due Date: October 20, 2023

1 Problems for Everyone

1. Write the following polynomials in the Taylor form, centered at $z = 2$.
   (a) $p(z) = z^5 + 3z + 4$
   (b) $p(z) = z^{10}$
   (c) $p(z) = (z-1)(z-2)^3$.
2. If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ ($a_n \neq 0$), then its reverse polynomial $p^*(z)$ is given by
$$p^*(z) = \overline{a_n} + \overline{a_{n-1}}z + \ldots + \overline{a_0}z^n.$$
   (a) Show that $p^*(z) = z^n\overline{p(1/\overline{z})}$.
   (b) Show that if $p(z)$ has a zero at $z_0 \neq 0$ then $p^*(z)$ has a zero at $1/\overline{z_0}$.
   (c) Show that for $|z| = 1$, we have $|p(z)| = |p^*(z)|$.
3. Let $f(z)$ be the rational function defined by
$$f(z) = \frac{2z + i}{(z^2 + z)(1-z)^2}.$$
   (a) Find all of the poles of this function and their multiplicities.
   (b) Find a partial fraction decomposition of this function.
   (c) If ζ is a pole of $f(z)$ then the coefficient of $\frac{1}{z-\zeta}$ in the partial fraction decomposition is called the residue of $f(z)$ at ζ and is denoted by $\mathrm{Res}(\zeta)$. Find the residues for all of the poles of this function.
4. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be the complex cosine function $f(z) = \cos(z)$.
   (a) Use the Cauchy-Riemann equations to show that $\cos(z)$ is an analytic function and prove that
$$\frac{df}{dz} = -\sin(z).$$
   (b) Compute the real and imaginary parts of the function $f(z^2)$.
   (c) Show that for $z \in \mathbb{C}, \cosh(z) = \cos(iz)$.
5. Prove that if $y \geq 0$ and $x \in \mathbb{R}$ then
$$|\cos(x + iy)| \leq e^y \text{ and } |\sin(x + iy)| \leq e^y.$$

6. Prove the following identities for $z \in \mathbb{C}$:
   (a) $\cosh^2(z) - \sinh^2(z) = 1$
   (b) $\cosh(z) = \cos(iz)$
   (c) $\sinh(z) = -i\sin(iz)$
   (d) $|\cosh(z)|^2 = \sinh^2(x) + \cos^2(y)$
   (e) $|\sinh(z)|^2 = \sinh^2(x) + \sin^2(y)$
   (f) $\overline{\sin(z)} = \sin(\overline{z})$

7. Show that if ξ is any value of
$$-i\log(iz + \sqrt{1-z^2})$$
then $\sin(\xi) = z$. Likewise, show that if ζ is any value of
$$\frac{i}{2}\log\left(\frac{1-iw}{1+iw}\right)$$
then $\tan(\zeta) = w$.

8. **Logarithms**
   (a) Write $\log(1-i)$ in the form $x+iy$, where $x, y \in \mathbb{R}$.
   (b) Write $\text{Log}(\sqrt{3}+i)$ in the form $x+iy$, where $x, y \in \mathbb{R}$.
   (c) Determine the domain of analyticity for $f(z) = \text{Log}(4+i-z)$.
   (d) Find all solutions $z \in \mathbb{C}$ to the equation $e^{2z} + e^z + 1 = 0$.

9. Find all the values of the given complex power
   (a) $(-1)^{3i}$
   (b) $3^{2i/\pi}$
   (c) $(1+i)^{1-i}$
   (d) $(1+\sqrt{3}i)^i$
   (e) $(-i)^i$
   (f) $(ei)^{\sqrt{2}}$

2 Graduate Problems

1. Prove first that
$$1 + e^{i\theta} + e^{2i\theta} + \ldots + e^{in\theta} = \frac{i}{2}\frac{\left(1 - e^{i(n+1)\theta}\right)e^{-i\theta/2}}{\sin(\theta/2)}.$$
Use this result to prove that
$$\frac{1}{2} + \cos(\theta) + \cos(2\theta) + \ldots + \cos(n\theta) = \frac{\sin\left((n+1/2)\theta\right)}{2\sin(\theta/2)}$$
and
$$\sin(\theta) + \sin(2\theta) + \ldots + \sin(n\theta) = \frac{\cos(\theta/2) - \cos((n+1/2)\theta)}{2\sin(\theta/2)}.$$