MTH 317/617
Homework #9

Due Date: December 1, 2023

1 Problems for Everyone

1. Let f be analytic except at an isolated singularity z_0 and suppose that the Laurent series for f about z_0 is given by
$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j.$$
Show that the coefficients a_j are given by
$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{j+1}} dz,$$
where Γ is any closed contour containing z_0.

2. Prove that if f has a simple pole at z_0 then
$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

3. Let $f(z) = P(z)/Q(z)$, where the functions $P(z)$ and $Q(z)$ are both analytic at z_0, and Q has a simple zero at z_0, while $P(z_0) \neq 0$. Prove that
$$\text{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

4. Prove that if f has a pole of order m at z_0, then
$$\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

5. Suppose that f is analytic and has a zero of order m at the point z_0. Show that the function $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with
$$\text{Res}(g; z_0) = m.$$

6. Determine all of the isolated singularities of each of the following functions and compute the residue at each singularity.

   (a) $f(z) = \dfrac{e^{3z}}{z-2}$

   (b) $f(z) = \dfrac{z+1}{z^2-3z+2}$

   (c) $f(z) = \dfrac{1+e^z}{z^2} + \dfrac{2}{z}$

   (d) $f(z) = \dfrac{\sin(z^2)}{z^2(z^2+1)}$

   (e) $f(z) = \dfrac{1-\cos(z)}{z^2}$

   (f) $f(z) = \dfrac{1}{z\sin(z)}$

   (g) $f(z) = \sin\left(\dfrac{1}{3z}\right)$

7. Evaluate the following contour integrals

   (a) $\displaystyle\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}dz$

   (b) $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z^6}dz$

   (c) $\displaystyle\int_{|z|=4} z\tan(z)dz$

   (d) $\displaystyle\int_{|z|=1} \frac{e^{z^2}}{z^6}dz$

   (e) $\displaystyle\int_{|z|=1} z^4\left(e^{z^{-1}}+z^2\right)dz$

   (f) $\displaystyle\int_{|z|=1} \cos\left(\frac{1}{z^2}\right)e^{z^{-1}}dz$

   (g) $\displaystyle\int_{|z|=1} \frac{1}{z^2(e^z-1)}dz$

8. Verify each of the following integrals by writing the integral as a contour integral in the complex plane:

   (a) $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin(\theta)}d\theta = \frac{2\pi}{\sqrt{3}}$

   (b) $\displaystyle\int_0^{2\pi} \frac{8}{5+2\cos(\theta)}d\theta = \frac{16\pi}{\sqrt{21}}$

   (c) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1+\sin^2(\theta)}d\theta = \sqrt{2}\pi$