Stochastic Calculus
Fall 2023
Exam 1
09/25/23

Name (Print): Key

This exam contains 7 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **You do not have to give explicit numbers for solutions to problems, i.e. you can leave the solution as a product and/or sum of numbers and you do not have to expand out terms with a factorial or binomial coefficients.**
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	15	
2	5	
3	5	
4	15	
5	5	
6	10	
7	15	
8	15	
9	15	
Total:	100	

Do not write in the table to the right.

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are independent then $A \cap B = \emptyset$.

   C **I** If $X : \Omega \mapsto \mathbb{R}$ is a random variable with corresponding probability density function $f(x)$, then for all $x \in \mathbb{R}$ it follows that $0 \leq f(x) \leq 1$.

   C **I** If $X : \Omega \mapsto \mathbb{R}$ is a random variable with corresponding cumulative distribution function $F_X(x)$, then $F_X(x)$ is a (not necessarily strictly) increasing function.

   C **I** If $X : \Omega \mapsto \mathbb{R}$ is a random variable with moment generating function $\phi(t)$, then $\phi(0) = 1$.

   C **I** If $X, Y : \Omega \mapsto \mathbb{R}$ are random variables then $\mathrm{Cov}(X, Y) \geq 0$.

2. (5 points) (**Short Answer**) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Write down the properties that P must satisfy by definition.

1. $P(\Omega) = 1$
2. $P(\phi) = 0$
3. If $A_1, A_2, \ldots$ are disjoint then
   $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$
4. For all $A \in \mathcal{F}$, $0 \leq P(A) \leq 1$.

3. (5 points) (**Short Answer**) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Write down the three properties that $\mathcal{F}$ must satisfy in order for $\mathcal{F}$ to be a σ-algebra.

1. $\Omega, \phi \in \mathcal{F}$
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $A_1 \cup A_2 \cup \ldots \in \mathcal{F}$.

4. (15 points) Let X be a random variable with the following density function

$$f(x) = \begin{cases} xe^{-x} & x > 0 \\ 0 & \leq 0 \end{cases}.$$

Find the probability density for the random variable

$$Y = X^3.$$

$$F_Y(y) = P(Y \leq y)$$
$$= P(X^3 \leq y)$$
$$= P(X \leq y^{1/3})$$
$$= \int_{-\infty}^{y^{1/3}} f(x)\,dx$$

$$\Rightarrow g(y) = \frac{dF_Y}{dy} = \frac{1}{3} y^{-2/3} f(y^{1/3})$$
$$= \begin{cases} \frac{1}{3} y^{-1/3} e^{-y^{1/3}}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

5. (5 points) (**Short Answer**) Let $X : \Omega \mapsto \mathbb{R}$ is a random variable. Assuming it exists, write down the formula for the moment generating function corresponding to X.

$$\phi(t) = \mathbb{E}[e^{tX}].$$

6. (10 points) Suppose $X : \Omega \mapsto \mathbb{R}$ is a random variable with moment generating function

$$\phi(t) = \left(1 - \frac{t}{2}\right)^{-3}.$$

(a) (5 points) Compute $\mathbb{E}[X]$.

$$\phi'(t) = \frac{3}{2}(1 - t/2)^{-4}$$

$$\Rightarrow \mathbb{E}[X] = \phi'(0) = 3/2$$

(b) (5 points) Compute $\mathrm{Var}[X]$.

$$\phi''(t) = \frac{12}{4}(1 - t/2)^{-5}$$

$$= 3(1 - t/2)$$

$$\Rightarrow \mathbb{E}[X^2] = \phi''(0) = 3.$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3 - 9/4 = 3/4.$$

7. (15 points) Let $X : \Omega \mapsto \mathbb{R}$ be a positive random variable with probability density $f(x)$.

(a) (5 points) (**Short Answer**) Write down Markhov's inequality for this random variable.

$$P(X > a) \leq \frac{1}{a}\mathbb{E}[X].$$

(b) (10 points) Use Markhov's inequality to prove that for all $a, \lambda > 0$

$$P(X > a) \leq \frac{1}{a^\lambda}\mathbb{E}[X^\lambda].$$

$$P(X > a) = P(X^\lambda > a^\lambda)$$
$$\leq \frac{1}{a^\lambda}\mathbb{E}[X^\lambda].$$

8. (15 points) Let $X, Y : \Omega \mapsto \mathbb{R}$ be random variables with following joint density

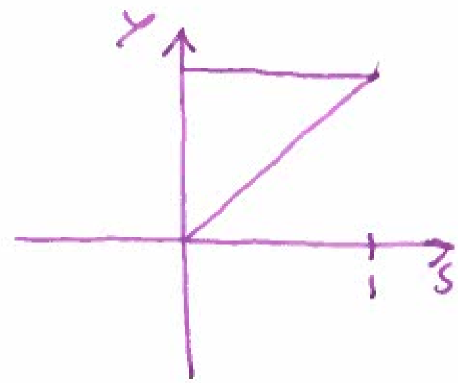
$$f(x,y) = \begin{cases} 8xy & 0 \leq x \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) (10 points) Compute the probability density of X and the probability density of Y.

$$P(X \leq x) = \int_0^x \int_s^1 8sy\,dy\,ds$$

$$= \int_0^x 4sy^2 \Big|_s^1 ds$$

$$\Rightarrow f(x) = 4x(1-x^2)$$



$$P(Y \leq y) = \int_0^y \int_0^s 8xs\,dx\,ds$$

$$= \int_0^y 4x^2 s \Big|_0^s ds$$

$$= \int_0^y 4s^3 ds$$

$$\Rightarrow g(y) = 4y^3$$

(b) (5 points) (**Short Answer**) Determine if X and Y are independent.

No, since $f(x,y) \neq f(x)g(y)$.

9. (15 points) Let X_1, X_2 be independent and identically distributed standard Gaussian random variables. Define the random variables Z_1 and Z_2 by

$$Z_1 = 2X_1 + 3X_2$$
$$Z_2 = 2X_1 - X_2.$$

(a) (5 points) Compute $\mathbb{E}[Z_1]$ and $\mathbb{E}[Z_2]$.

$\mathbb{E}[Z_1] = 2\mathbb{E}[X_1] + 3\mathbb{E}[X_2] = 0$

$\mathbb{E}[Z_2] = 2\mathbb{E}[X_1] - \mathbb{E}[X_2] = 0$

(b) (5 points) Compute $\mathrm{Var}[Z_1]$ and $\mathrm{Var}[Z_2]$.

$\mathrm{Var}[Z_1] = \mathrm{Cov}(Z_1, Z_1)$

$= 4\,\mathrm{Cov}(X_1, X_1) + 9\,\mathrm{Cov}(X_2, X_2) + 12\,\mathrm{Cov}(X_1, X_2)$

$= 13$

$\mathrm{Var}[Z_2] = 4\,\mathrm{Cov}(X_1, X_1) + \mathrm{Cov}(X_2, X_2)$

$= 5$

(c) (5 points) Compute $\mathrm{Cov}(Z_1, Z_2)$.

$\mathrm{Cov}(Z_1, Z_2) = 4\,\mathrm{Cov}(X_1, X_1) - 3\,\mathrm{Cov}(X_2, X_2)$

$= 1.$