MTH 383/683: Homework #1

Due Date: September 08, 2023

1 Problems for Everyone

1. **Basic properties of probability**. Let P be a probability on Ω. Use the basic properties of probability to prove the following

   (a) *Finite Additivity*: If A, B are disjoint events then $P(A \cup B) = P(A) + P(B)$.

   (b) For any event A, $P(A^c) = 1 - P(A)$.

   (c) For any events A, B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   (d) *Monotinicity*: If $A \subset B$, $P(A) \leq P(B)$.

2. **Distribution as a probability on $\mathbb{R}$**. Let ρ_X be the distribution of a random variable X on some probability space $(\Omega, \mathcal{F}, P)$. Show that ρ_X has the properties of a probability distribution on $\mathbb{R}$.

3. **Distribution of an indicator function**. Let $(\Omega, \mathcal{F}, P)$ be a probability space and A and event in $\mathcal{F}$ with $0 < P(A) < 1$. What is the distribution of the random variable $\mathbb{1}_A$?

4. **Constructing a random variable from another one**. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1, 1]$. Consider $Y = X^2$.

   (a) Find the CDF of Y and plot its graph.

   (b) Find the PDF of Y and plot its graph.

5. **Memory loss property**. Let Y be an exponential random variable with parameter $\lambda > 0$. Show that for any $s, t > 0$

   $$P(Y > t + s | Y > s) = P(Y > t).$$

   Recall that the conditional probability of A given the event B is

   $$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

6. **Independence** Recall that two events A and B in some probability space $(\Omega, \mathcal{F}, P)$ are independent if

   $$P(A \cap B) = P(A)P(B).$$

   Consequently, if A and B are independent it follows that $P(A|B) = P(A)$ as expected.

   (a) Prove that if A and B are independent than A^c and B are also independent.

   (b) Prove that if A and B are independent then A^c and B^c are also independent.

   (c) Prove that if A and B are independent then $P(A \cup B) = 1 - (1 - P(A))(1 - P(B))$.

2 Graduate Problems (undergraduates can complete for extra credit but your homework score cannot go above 10 points)

1. Suppose $A_1, A_2, \ldots$ are events in a probability space $(\Omega, \mathcal{F}, P)$.

   (a) Prove that

   $$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{\omega \in \Omega : \omega \text{ belongs to infinitely many } A_n\}.$$

   The event $\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ is called "A_n infinitely often" and is abbreviated "A_n i.o.".

   (b) Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then

   $$P(A_n \text{ i.o.}) = 0.$$

   (c) $A_1, A_2, \ldots$ are called mutually independent if every A_i is independent of any intersection of the other A_j for $j \neq i$, that is, for every finite subsequence A_{j_k} of events

   $$P\left(\bigcap_k A_{j_k}\right) = \prod_k P(A_{j_k}).$$

   Prove that if $A_1, A_2, \ldots$ are mutually independent events then

   $$P\left(\bigcup_{k=n}^{\infty} A_k\right) = 1 - \prod_{k=n}^{\infty} \left(1 - P(A_k)\right).$$

   (d) Prove that for all $x \in \mathbb{R}$, $1 - x \leq e^{-x}$ and use this to prove that for mutually independent events $A_1, A_2, \ldots$ it follows that

   $$P\left(\bigcup_{k=n}^{\infty} A_k\right) \geq 1 - \prod_{k=n}^{\infty} e^{-P(A_k)} = 1 - e^{\sum_{k=n}^{\infty} A_k}.$$

   (e) If $A_1, A_2, \ldots$ are mutually independent events, prove that if $\sum_{n=1}^{\infty} P(A_n) = \infty$ then

   $$P(A_n \text{ i.o.}) = 1.$$

   (f) Suppose that the events $A_1, A_2, \ldots$ are mutually independent with

   $$P\left(\bigcup_n A_n\right) = 1 \text{ and } P(A_n) < 1$$

   for each n. Prove that $P(A_n \text{ i.o.}) = 1$.

2. Let Ω be any set and $\mathcal{A}$ any collection of subsets of Ω. Show that there exists a unique smallest σ-algebra $\mathcal{F}$ of subsets of Ω containing $\mathcal{A}$. We call $\mathcal{F}$ the σ-algebra generated by $\mathcal{A}$. **Hint:** Consider the intersection of all σ-algebras containing $\mathcal{A}$.

Homework #1

#1 : Basic Properties of Probability

Let P be a probability on Ω. Prove the following

(a) If $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$

(b) $P(A^c) = 1 - P(A)$.

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) If $A \subset B$, $P(A) \leq P(B)$

Solution:

(a)
$$\begin{aligned} P(A \cup B) &= P(A \cup B \cup \phi \cup \phi \cup \cdots) \\ &= P(A) + P(B) + P(\phi) + P(\phi) + \cdots \\ &= P(A) + P(B). \end{aligned}$$

(b)
$$\begin{aligned} 1 &= P(\Omega) \\ &= P(A \cup A^c) \\ &= P(A) + P(A^c). \end{aligned}$$

Therefore, $1 - P(A) = P(A^c)$.

(c) Computing we have that

$$P(A) = P((B^c \cap A) \cup (A \cap B)) = P(B^c \cap A) + P(A \cap B)$$

$$P(B) = P((B \cap A^c) \cup (A \cap B)) = P(B \cap A^c) + P(A \cap B)$$

$$P(A \cup B) = P((B^c \cap A) \cup (A^c \cap B) \cup (A \cap B)) = P(B^c \cap A) + P(A^c \cap B) + P(A \cap B$$

Therefore,

$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(B^c \cap A) + P(B \cap A^c) + P(A \cap B) \\ &= P(A \cup B). \end{aligned}$$

(d) $P(B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$.

Therefore, $P(A) \leq P(B)$.

$\blacksquare$

#3: Distribution of an Indicator Function

Let $(\Omega, \mathcal{F}, P)$ be a probability space and A an event in $\mathcal{F}$ with $0 < P(A) < 1$. What is the distribution of the random variable $\mathbb{1}_A$?

Solution:

If we let $X = \mathbb{1}_A$ then

$$\begin{aligned} \mathcal{S}_X((a,b)) &= P(X \in (a,b)) \\ &= P(\{\omega \in \Omega : \mathbb{1}_A(\omega) \in (a,b)\}) \\ &= \begin{cases} 0, & \text{if } a<b<0 \text{ or } 1<a<b \\ 1-P(A) & \text{if } a<0<b<1 \\ P(A) & \text{if } 0<a<1<b \\ 1 & \text{if } a<0 \text{ and } b>1 \end{cases} \end{aligned}$$

#4. Constructing a Random Variable from Another One

Let X a random variable on $(\Omega, \mathcal{F}, P)$ that is is uniformly distributed on $[-1,1]$. Consider $Y = X^2$.

(a) Find the CDF of Y and plot its graph.

(b) Find the PDF of Y and plot its graph.

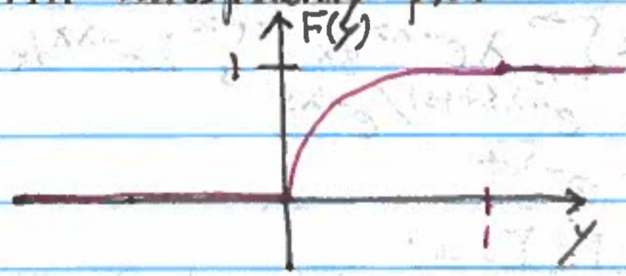
Solution:

(a) $$\begin{aligned} P(Y \le y) &= P(X^2 \le y) \\ &= \begin{cases} 0, & \text{if } y \le 0 \\ \frac{1}{2}\int_{-\sqrt{y}}^{\sqrt{y}} 1\,ds, & \text{if } 0 \le y \le 1 \\ 1, & \text{if } y > 1 \end{cases} \end{aligned}$$

Therefore,

$$F(y) = \begin{cases} 0, & \text{if } y \leq 0 \\ \sqrt{y}, & \text{if } 0 \leq y \leq 1 \\ 1, & \text{if } y > 1 \end{cases}$$

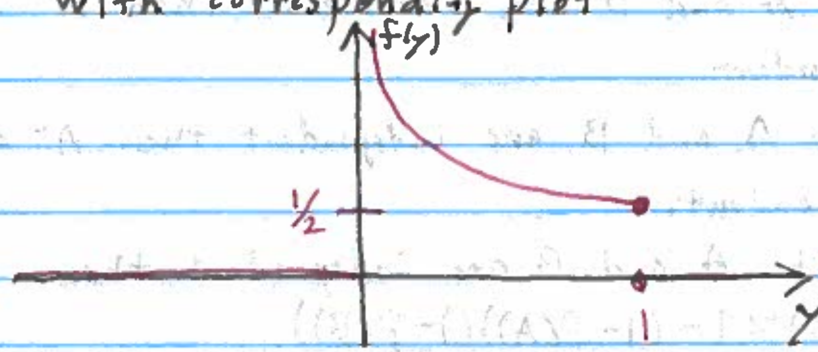
with corresponding plot



(b). The PDF is therefore given by

$$f(y) = \frac{dF}{dy} = \begin{cases} 0, & \text{if } y < 0 \text{ or } y > 1 \\ \frac{1}{2} y^{-1/2}, & \text{if } 0 < y < 1 \end{cases}$$

with corresponding plot



∎

#5: Memory Loss Problem

Let X be an exponential random variable with parameter $\lambda > 0$. Show that for any $s, t > 0$

$$P(X \geq t+s \mid X > s) = P(X > t).$$

proof:

$$
\begin{aligned}
P(X \geq t+s \mid X > s) &= \int_{t+s}^{\infty} \lambda e^{-\lambda x}\,dx \Big/ \int_{s}^{\infty} \lambda e^{-\lambda x}\,dx \\
&= e^{-\lambda(t+s)} / e^{-\lambda s} \\
&= e^{-\lambda t} \\
&= P(X > t).
\end{aligned}
$$

#6: Independence

Recall that two events A, B in some probability space are independent if

$$P(A \cap B) = P(A)P(B).$$

(a) Prove that if A and B are independent then A^c and B are also independent.

(b) Prove that if A and B are independent then A^c and B^c are also independent.

(c) Prove that if A and B are independent then

$$P(A \cup B) = 1 - (1 - P(A))(1 - P(B)).$$

Solution:

(a). Computing, we have that

$$
\begin{aligned}
P(B) &= P(\Omega \cap B) \\
&= P((A^c \cup A) \cap B) \\
&= P((A^c \cap B) \cup (A \cap B)) \\
&= P(A^c \cap B) + P(A \cap B)
\end{aligned}
$$

$\Rightarrow P(B) = P(A^c \cap B) + P(A)P(B)$

$\Rightarrow P(B)(1-P(A)) = P(A^c \cap B)$

$\Rightarrow P(A^c \cap B) = P(B)P(A^c)$.

(b) $P(A^c \cap B^c) = P((A \cup B)^c)$

$$= 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= P(A^c) + P(B)(P(A) - 1)$$

$$= P(A^c) - P(B)P(A^c)$$

$$= P(A^c)(1 - P(B))$$

$$= P(A^c)P(B^c)$$

(c) $P(A \cup B) = P((A^c \cap B^c)^c)$

$$= 1 - P(A^c \cap B^c)$$

$$= 1 - P(A^c)P(B^c)$$

$$= 1 - (1 - P(A))(1 - P(B))$$

■

Graduate Problem #1.

Solution:

(b). For all $n \in \mathbb{N}$ it follows that

$$P(A_n \text{ i.o.}) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m \geq n}^{\infty} A_m\right)$$

$$\leq P\left(\bigcup_{m \geq n}^{\infty} A_m\right)$$

$$\leq \sum_{m \geq n}^{\infty} P(A_m).$$

Since $\sum_{n=1}^{\infty} P(A_n) < \infty$ it follows that $\lim_{n \to \infty} \sum_{m \geq n}^{\infty} P(A_m) = 0$ and thus $P(A_n \text{ i.o.}) = 0$.

(c) Since $A_1, A_2, \ldots$ are mutually independent it follows that

$$P\left(\bigcup_{k=n}^{\infty} A_k\right) = P\left(\left(\bigcap_{k=n}^{\infty} A_k^c\right)^c\right)$$
$$= 1 - P\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$
$$= 1 - \prod_{k=n}^{\infty} P(A_k^c)$$
$$= 1 - \prod_{k=n}^{\infty} (1 - P(A_k)).$$

(d) It follows from part (c) that

$$P\left(\bigcup_{k=n}^{\infty} A_k\right) = 1 - \prod_{k=n}^{\infty}(1 - P(A_k))$$
$$\geq 1 - \prod_{k=n}^{\infty} e^{-P(A_k)}$$
$$= 1 - e^{-\sum P(A_k)}$$
$$= 1$$

Therefore,

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_k\right) = 1.$$

(e) $1 = P\left(\bigcup_{n=1}^{\infty} A_n\right) \geq 1 - e^{-\sum_{n=1}^{\infty} P(A_n)}$ and thus $\sum_{n=1}^{\infty} P(A_n) = \infty$ proving that $P(A_n \text{ i.o.}) = 1$.

#2.

Let $\mathcal{F}_\alpha$ denote the family of σ-algebras containing $\mathcal{A}$ and define $\mathcal{F} = \bigcap_\alpha \mathcal{F}_\alpha$.

1. Since $\Omega, \phi \in \mathcal{F}_\alpha$ for all α it follows that $\Omega, \phi \in \bigcap_\alpha \mathcal{F}_\alpha$.
2. If $A \in \mathcal{F}$ then for all α, $A^c \in \mathcal{F}_\alpha$ and thus $A^c \in \bigcap_\alpha \mathcal{F}_\alpha = \mathcal{F}$.
3. If $A_1, \ldots, A_n \in \mathcal{F}$ then for all α, $A_1, \ldots, A_n \in \mathcal{F}_\alpha$ and thus for all α $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{F}_\alpha$. Therefore, $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{F} = \bigcap_\alpha \mathcal{F}_\alpha$.

Suppose there exists another σ-algebra $\mathcal{G}$ for which $\mathcal{A} \in \mathcal{G}$. Therefore, there exists α for which $\mathcal{F}_\alpha = \mathcal{G}$ and thus $\mathcal{F} \subseteq \mathcal{G}$.

∎