MTH 383/683: Homework #3

Due Date: September 22, 2023

1 Problems for Everyone

1. **Sum of Exponential is Gamma.** Let $X_1, \ldots, X_n$ be n independent and identically distributed exponential random variables with parameter $\lambda > 0$ and let $Z = X_1 + \ldots + X_n$. Recall from the last homework that the moment generating function of each X_i is given by

$$\phi(t) = \mathbb{E}[e^{tX_i}] = \frac{\lambda}{\lambda - t}.$$

for $t < \lambda$.

   (a) Find the moment generating function of Z.

   (b) A random variable Y is said to have a gamma distribution with parameter $\lambda > 0$ if it has the following probability density

$$f(y) = \begin{cases} \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases},$$

   where $n \in \mathbb{N}$. Find the moment generating function Y and use this result to prove that the sum of n independent and identically distributed exponential random variables is a gamma distribution.

2. **Sum of Gaussian is Gaussian** Let X_1, X_2 be two independent but not necessarily identically distributed Gaussian random variables. Recall from last homework that the moment generating function of a Gaussian random variable with mean μ and standard deviation σ is given by

$$\phi(t) = \mathbb{E}[e^{tX}] = e^{t\mu + t^2\sigma^2/2}.$$

   By computing its moment generating function, prove that $Z = X_1 + X_2$ is also a Gaussian random variable.

3. **Calculations with Joint Density** Let $\vec{X} = (X, Y)$ be a random vector with joint density

$$f(x, y) = \begin{cases} 6x^2y & 0 \leq x \leq y \text{ and } x + y \leq 2 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Find the probability density functions of X and Y.

   (b) Are X and Y independent random variables?

4. **Example of Uncorrelated Random Variables that are not Independent** Let X be a standard Gaussian. Show that $\text{Cov}(X, X^2) = 0$, yet X and X^2 are not independent.

5. **The Covariance Matrix of a Random Vector is Always Positive Definite.** Let C be the covariance matrix of the random vector $X = (X_1, \dots X_n)$. Show that C is always positive semidefinite, i.e.,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j C_{ij} \geq 0.$$

for any $a_1, \dots a_n \in \mathbb{R}^n$. **Hint:** Write the left side as the variance of some random variable.

6. **A Linear Transformation of a Gaussian Vector is also Gaussian.** Let $X = (X_1, \dots X_n)$ be an n-dimensional Gaussian vector and M a $m \times n$ matrix.

   (a) Show that $Y = MX$ is also a Gaussian vector.

   (b) If the covariance matrix of X is C, write the covariance matrix of Y in terms of M and C.

Homework #3

#1

Prove that the sum of I.I.D. exponentially distributed random variables has a Γ-distribution.

Solution:

Let $X_1, \ldots, X_n$ have exponential distributions with parameter $\lambda > 0$. Therefore,

$$\begin{aligned}
\phi(t) &= \mathbb{E}[e^{t(X_1 + \cdots + X_n)}] \\
&= \mathbb{E}[e^{tX_1 + \cdots + tX_n}] \\
&= \mathbb{E}[e^{tX_1} \cdots e^{tX_n}] \\
&= \mathbb{E}[e^{tX_1}] \cdots \mathbb{E}[e^{tX_n}] \\
&= \frac{\lambda}{\lambda - t} \cdots \frac{\lambda}{\lambda - t} \\
&= \frac{\lambda^n}{(\lambda - t)^n}.
\end{aligned}$$

Now, if we let Y be a Γ-distribution it follows that

$$\begin{aligned}
\mathbb{E}[e^{tY}] &= \frac{\lambda^n}{(n-1)!} \int_0^\infty e^{ty} y^{n-1} e^{-\lambda y}\, dy \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty y^{n-1} e^{-(\lambda - t)y}\, dy \\
&= \frac{\lambda^n}{(n-1)!} \left(\frac{-1}{\lambda - t} y^{n-1} e^{-(\lambda - t)y} \Big|_0^\infty + \int_0^\infty \frac{n-1}{\lambda - t} y^{n-2} e^{-(\lambda - t)y}\, dy \right) \\
&= \frac{\lambda^n}{(n-2)!} \frac{1}{(\lambda - t)} \int_0^\infty y^{n-2} e^{-(\lambda - t)y}\, dy
\end{aligned}$$

Inductively, it follows that

$$\mathbb{E}[e^{tY}] = \frac{\lambda^n}{(\lambda - t)^n} = \phi(t)$$

#2.

Let X_1, X_2 be two independent Gaussian random variables. Prove that $Z = X_1 + X_2$ is a Gaussian random variable.

Solution:

Computing, we have that

$$\begin{aligned}
\phi(t) &= \mathbb{E}[e^{tZ}] \\
&= \mathbb{E}[e^{t(X_1+X_2)}] \\
&= \mathbb{E}[e^{tX_1} e^{tX_2}] \\
&= \mathbb{E}[e^{tX_1}]\,\mathbb{E}[e^{tX_2}] \\
&= e^{t\mu_1 + t^2\sigma_1^2/2} e^{t\mu_2 + t^2\sigma_2^2/2} \\
&= e^{t(\mu_1+\mu_2) + (\sigma_1^2+\sigma_2^2)/2\, t^2},
\end{aligned}$$

Therefore, Z_1 is a Gaussian random variable with mean $\mu = \mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

$\blacksquare$

#3.

Let $\vec{X} = (X, Y)$ be a random vector with joint density

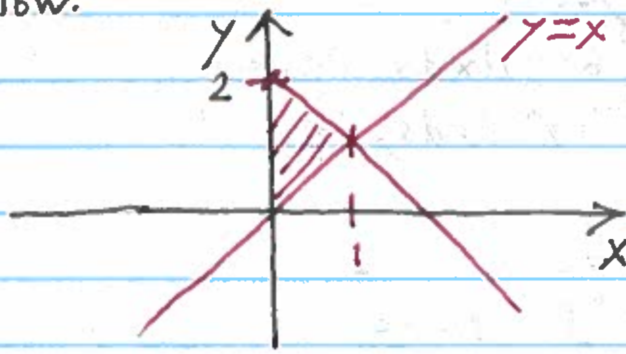
$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y \text{ and } x+y \le 2, \\ 0, & \text{e.w.} \end{cases}$$

(a) Find the probability density functions of X and Y.

(b) Are X and Y independent random variables?

Solution:

(a) The regions in which $f(x,y) \geq 0$ are plotted below:



Therefore,

$$P(X \leq x) = \begin{cases} 0, & \text{if } x < 0 \\ \int_0^x \int_s^{2-s} 6 s^2 y \, dy \, ds & \text{if } 0 < x < 1 \\ 1, & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} 0, & \text{if } x < 0 \\ \int_0^x 3 s^2 y^2 \Big|_s^{2-s} ds & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} 0, & \text{if } x < 0 \\ \int_0^x [3(2-s)^2 s^2 - 3 s^4] ds & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$$\Rightarrow f(x) = \frac{d P(X \leq x)}{dx} = 3 s^2 [4 - 4s + s^2 - s^2]$$

$$\Rightarrow f(x) = 12 s^3 [1 - s].$$

We also have that

$$P(Y \leq y) = \begin{cases} 0, & \text{if } y < 0 \\ \int_0^y \int_0^s 6x^2 s \, dx \, ds, & 0 < y < 1 \\ \int_0^1 \int_0^s 6x^2 s \, dx \, ds + \int_1^y \int_0^{2-s} 6x^2 s \, dx \, ds, & 1 < y < 2 \\ 1, & y \geq 2 \end{cases}$$

$$\Rightarrow g(y) = \frac{d\,P(Y \leq y)}{dy}$$

$$= \begin{cases} 0, & y<0 \text{ or } y>2 \\ \frac{d}{dy}\int_0^y \int_0^s 6x^2 s\,dx\,ds, & 0<y<1 \\ \frac{d}{dy}\int_1^y \int_0^{2-s} 6x^2 s\,dx\,ds, & 1<y<2 \end{cases}$$

$$= \begin{cases} 0, & y<0 \text{ or } y>2 \\ \frac{d}{dy}\int_0^y 2s^4\,ds, & 0<y<1 \\ \frac{d}{dy}\int_1^y 2(2-s)^3 s, & 1<y<2 \end{cases}$$

$$\Rightarrow g(y) = \begin{cases} 0, & y<0 \text{ or } y>2 \\ 2y^4, & 0<y<1 \\ 2(2-y)^3 y, & 1<y<2 \end{cases}$$

Since $f(x,y) \neq f(x)g(y)$ it follows that X, Y are not independent. ∎

#4.

Let X be a standard Gaussian. Show that $Cov(X, X^2) = 0$ yet X and X^2 are not independent.

Solution:

Computing, we have that $Cov(X, X^2) = \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] = 0$ since

$$\mathbb{E}[X^3] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^3 e^{-x^2/\sigma^2} dx = 0$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-x^2/\sigma^2} dx = 0$$

However,

$$P(0<X<1,\ 0<X^2<1)=P(0<X<1),$$

$$P(0<X<1)P(0<X^2<1)=P(0<X<1)P(-1<X<1).$$

Therefore, if these random variables are independent we have that

$$P(-1<X<1)=1$$

which is a contradiction.

∎

#5

Let C be the covariance matrix of $(X_1,\dots,X_n)$. Show that C is always positive semidefinite.

Solution:

For $a_1,\dots,a_n\in\mathbb{R}$ we have that

$$Var[a_1X_1+\dots+a_nX_n]\geq 0$$

$$\Rightarrow Cov(a_1X_1+\dots+a_nX_n,\ a_1X_1+\dots+a_nX_n)=\sum_{i\geq1}^{n}\sum_{j\geq1}^{n}a_ia_jC_{ij}\geq 0.$$

∎

#6.

Let $X=(X_1,\dots,X_n)$ be a Gaussian vector and M a $m\times n$ matrix.

(a) Show that $Y=MX$ is also Gaussian

(b) If the covariance matrix of X is C, write the covariance matrix of Y in terms of M and C.

Solution:

(a) If we conder a linear combination of $(Y_1, \ldots, Y_m)$ we have that

$$a_1 Y_1 + \ldots + a_m Y_m = a_1 \sum_{i=1}^{n} M_{1i} X_i + \ldots + a_m \sum_{i=1}^{n} M_{mi} X_i$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} a_j M_{ji} X_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_j M_{ji} \right) X_i$$

$$= \left(\sum_{j=1}^{m} a_j M_{j1} \right) X_1 + \ldots + \left(\sum_{j=1}^{m} a_j M_{jn} \right) X_n$$

and thus $a Y_1 + \ldots + a_m Y_m$ is a Gaussian random variable.

(b) Computing, we have that

$$Cov(Y_i, Y_j) = Cov\left(\sum_{k=1}^{n} M_{ik} X_k, \sum_{l=1}^{n} M_{jl} X_l \right)$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} M_{ik} M_{jl} Cov(X_k, X_l)$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} M_{ik} M_{jl} C_{kl}$$

Therefore, if let C' denote the covariance matrix of Y it follows that

$$C' = \underset{m\times n}{M} \underset{n\times n}{C} \underset{n\times m}{M^T}$$