MTH 383/683: Homework #5

Due Date: October 20, 2023

1 Problems for Everyone

2 1. **Triangle Inequality** Let X, Y be two random variables in $L^2(\Omega, \mathcal{F}, P)$. Prove the triangle inequality

$$\|X + Y\| \leq \|X\| + \|Y\|.$$

Hint: Write out the norms as expectations, expand, and use the Cauchy-Schwarz inequality. Use this result to prove the more general inequality: for $X_1, \ldots X_n \in L^2(\Omega, \mathcal{F}, P)$ we have

$$\|X_1 + \ldots X_n\| \leq \sum_{j=1}^{n} \|X_i\|.$$

2 2. **Integrable Random Variables** Recall that $L^1(\Omega, \mathcal{F}, P)$ is the set of all random variables for which $\mathbb{E}[|X|] < \infty$.

(a) Prove that $L^1(\Omega, \mathcal{F}, P)$ is a linear subspace of the vector space of random variables.

(b) Verify that $\|X\|_1 = \mathbb{E}[|X|]$ is a norm for L^1. That is prove the following

i. For all $a \in \mathbb{R}$ and $X \in L^1$ that $\|aX\|_1 = |a|\|X\|_{L^1}$.

ii. For all $X \in L^1$, $\|X\|_{L^1} = 0$ if and only if $X = 0$.

iii. For all $X, Y \in L^1$, $\|X + Y\| \leq \|X\| + \|Y\|$.

2 3. **Brownian Moments** Let B_t be standard Brownian motion. Compute the following moments:

(a) $\mathbb{E}[B_t^6]$

(b) $\mathbb{E}[(B_{t_2} - B_{t_1})(B_{t_3} - B_{t_2})]$ if $t_1 < t_2 < t_3$.

(c) $\mathbb{E}[B_s^2 B_t^2]$ if $s < t$

(d) $\mathbb{E}[B_s B_t^3]$ if $s < t$

(e) $\mathbb{E}[B_s^{100} B_t^{101}]$

2 4. **Brownian Probabilities** Let B_t be a standard Brownian motion. Write the following probabilities as an integral.

(a) $P(B_1 > 1, B_2 > 1)$

(b) $P(B_1 > 1, B_2 > 1, B_3 > 1)$

2 5. **Time Inversion** Let B_t be a standard Brownian motion and $X_t = tB_{1/t}$ for $t > 0$.

(a) Show that X_t has the distribution of a Brownian motion on $t > 0$.

(b) Prove that X_t converges to 0 as $t \to 0$ in the sense of L^2 convergence.

(c) Prove the following law of large numbers for Brownian motion

$$\lim_{t \to \infty} \frac{X_t}{t} = 0.$$

Homework #5

#1: Triangle Inequality

Let X, Y be two random variables in $L^2(\Omega, \mathcal{F}, P)$. Prove the triangle inequality.

~~proof~~:

Expanding, we have that

$$\|X+Y\|^2 = \langle X+Y, X+Y\rangle$$
$$= \langle X,X\rangle + 2\langle X,Y\rangle + \langle Y,Y\rangle$$
$$= \|X\|^2 + 2\langle X,Y\rangle + \|Y\|^2$$
$$\leq \|X\|^2 + 2|\langle X,Y\rangle| + \|Y\|^2$$
$$\leq \|X\|^2 + 2\|X\|\cdot\|Y\| + \|Y\|^2$$
$$= (\|X\| + \|Y\|)^2$$
$$\Rightarrow \|X+Y\| \leq \|X\| + \|Y\|.$$

Furthermore, for $X_1, \ldots, X_n \in L^2(\Omega, \mathcal{F}, P)$ we have that

$$\|X_1 + X_2 + \ldots + X_n\| \leq \|X_1\| + \|X_2 + X_3 + \ldots + X_n\|$$
$$\leq \|X_1\| + \|X_2\| + \|X_3 + \ldots + X_n\|$$
$$\vdots$$
$$\leq \|X_1\| + \|X_2\| + \ldots + \|X_n\|$$

∎

#3: Brownian Moments

(a) $\mathbb{E}[B_t^6]$

(b) $\mathbb{E}[(B_{t_2}-B_{t_1})(B_{t_3}-B_{t_2})]$ $\quad t_1<t_2<t_3$

(c) $\mathbb{E}[B_s^2 B_t^2]$, $s<t$

(d) $\mathbb{E}[B_s B_t^3]$, $s<t$

(e) $\mathbb{E}[B_s^{100} B_t^{101}]$, $s<t$

Solution:

(a)
$$\mathbb{E}[B_t^6] = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} x^6 e^{-x^2/2t^2}dx$$
$$= \sqrt{\tfrac{2}{\pi}}\cdot\tfrac{1}{t}\int_0^{\infty} x^6 e^{-x^2/2t^2}dx$$
$$= 15t^3$$

(b)
$$\mathbb{E}[(B_{t_2}-B_{t_1})(B_{t_3}-B_{t_2})] = \mathbb{E}[B_{t_2}-B_{t_1}]\,\mathbb{E}[B_{t_3}-B_{t_2}]$$
$$= 0\cdot 0$$
$$= 0$$

(c)
$$\mathbb{E}[B_s^2 B_t^2] = \mathbb{E}[B_s^2(B_t-B_s+B_s)^2]$$
$$= \mathbb{E}[B_s^2[(B_t-B_s)^2+2B_s(B_t-B_s)+B_s^2]]$$
$$= \mathbb{E}[B_s^2(B_t-B_s)^2]+2\,\mathbb{E}[B_s^3]\,\mathbb{E}[B_t-B_s]+\mathbb{E}[B_s^4]$$
$$= s(t-s)+0+3s^2$$
$$= 3s^2+s(t-s)$$

(d)
$$\mathbb{E}[B_s B_t^3] = \mathbb{E}[B_s(B_t-B_s+B_s)^3]$$
$$= \mathbb{E}[B_s((B_t-B_s)^3+3(B_t-B_s)^2B_s+3(B_t-B_s)B_s^2+B_s^3)]$$
$$= \mathbb{E}[B_s]\,\mathbb{E}[(B_t-B_s)^3]+3\,\mathbb{E}[B_s^2]\,\mathbb{E}[(B_t-B_s)^2]+3\,\mathbb{E}[B_s^3]\,\mathbb{E}[B_t-B_s]+\mathbb{E}[B_s^4]$$
$$= 3s(t-s)+3s^2$$

(e)
$$\mathbb{E}[B_s^{100}B_t^{101}] = \mathbb{E}[B_s^{100}(B_t-B_s+B_s)^{101}]$$
$$= \mathbb{E}[B_s^{100}((B_t-B_s)^{101}+c_1(B_t-B_s)^{100}B_s+c_2(B_t-B_s)^{99}B_s^2+\dots+c_{101}B_s^{101}]$$
$$= \mathbb{E}[B_s^{100}]\,\mathbb{E}[(B_t-B_s)^{101}]+c_1\mathbb{E}[B_s^{101}]\,\mathbb{E}[(B_t-B_s)^{100}]+c_2\,\mathbb{E}[B_s^{102}]\,\mathbb{E}[(B_t-B_s)^{99}]$$
$$+\dots+c_{101}\,\mathbb{E}[B_s^{201}]$$
$$= 0+0+0+\dots+0=0. \quad\blacksquare$$

#4: Brownian Probabilities

Let B_t be a standard Brownian motion. Write the following probabilities as an integral.

(a) $P(B_1 > 1, B_2 > 1)$

(b) $P(B_1 > 1, B_2 > 1, B_3 > 1)$.

Solution:

(a) For B_1, B_2 the covariance matrix is given by

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow C^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow p(x_1, x_2) = \frac{1}{\sqrt{2\pi t}} e^{-\langle (x_1, x_2), (2x_1 - x_2, -x_1 + x_2) \rangle / 2}$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-2x_1^2 + x_1 x_2 + x_1 x_2 - x_2^2 / 2}$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-2x_1^2 + 2x_1 x_2 - x_2^2 / 2}$$

(b.) For B_1, B_2, B_3 the covariance matrix is given by

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow \det(C) = 2 - 1 + 0 = 1$$

Inverting, we have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R1 \\ -R1 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -R2 \\ \\ -R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -R3 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Consequently,

$$[x_1\ x_2\ x_3]C^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=[x_1\ x_2\ x_3]\begin{bmatrix}2x_1-x_2\\-x_1+2x_2-x_3\\-x_2+x_3\end{bmatrix}=2x_1^2-x_1x_2-x_1x_2+2x_2^2-x_2x_3-x_2x_3+x_3^2$$

Therefore,

$$P(B_1>1, B_2>1, B_3>1)=\frac{1}{(2\pi)^{3/2}}\int_1^\infty\int_1^\infty\int_1^\infty e^{-(2x_1^2-2x_1x_2+2x_2^2-2x_2x_3+x_3^2)/2}\,dx_1dx_2dx_3$$

■

#5: Time Inversion

Let B_t be a standard Brownian motion and $X_t=tB_{1/t}$.

(a) Show that X_t has the distribution of a Brownian motion.

(b) Prove that X_t converges to 0 as $t\to 0$ in the sense of L^2.

(c) Prove the following law of large numbers for Brownian motion

$$\lim_{t\to\infty} X_t/t=0.$$

Solution:

(a) Assuming $s<t$ we have

$$\begin{aligned}\langle X_t, X_s\rangle&=\langle tB_{1/t}, sB_{1/s}\rangle\\&=ts\langle B_{1/t}, B_{1/s}\rangle\\&=ts\min\{1/t,1/s\}\\&=ts/t\\&=s\end{aligned}$$

Therefore, X_t is a Brownian motion.

(b) Computing, we have that $\|X_t\|^2=t$ and thus $X_t\xrightarrow{L^2}0$.

(c) Computing, we have that

$$\lim_{t\to\infty} X_t/t=\lim_{t\to\infty} B_{1/t}=B_0=0.$$

■