MTH 383/683: Homework #2

Due Date: September 15, 2023

1 Problems for Everyone

1. **Gaussian Integration by Parts**. Let Z be a standard Gaussian random variable.

   (a) Show using integration by parts that for a differentiable function g,

   $$\mathbb{E}[Zg(Z)] = \mathbb{E}[g'(Z)].$$

   (b) Use this result to prove that for $j \in \mathbb{N}$,

   $$\mathbb{E}\left[Z^{2j}\right] = \frac{(2j)!}{2^j j!} = (2j-1)(2j-3)\cdots 5\cdot 3\cdot 1.$$

2. **MGF of Exponential Random Variables** Let X be a random variable with an exponential distribution with parameter $\lambda > 0$.

   (a) Show that the MGF of X is given by

   $$\phi(t) = \mathbb{E}\left[e^{tX}\right] = \frac{\lambda}{\lambda - t},\ t < \lambda.$$

   (b) Use $\phi(t)$ to compute the expectation and variance of X.

3. **Guassian Tail**. Consider a random variable X with finite MGF such that

   $$\phi(t) = \mathbb{E}\left[e^{\lambda X}\right] \leq e^{t^2/2}$$

   for $\lambda > 0$. Using Chernoff's bound, prove that for $a > 0$,

   $$P(X > a) \leq e^{-a^2/2}.$$

4. **Constructing a Random Variable from Another One**. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1, 1]$. Find a function $f : [-1, 1] \mapsto \mathbb{R}^+$ such that $Y = f(X)$ has an exponential distribution with parameter $\lambda > 0$.

5. **Why $\sqrt{2\pi}$?** Use polar coordinates to prove that

   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dxdy = \pi.$$

   Conclude that this implies that

   $$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 1.$$

6. **Gaussian Random Variables**. Let Z be a standard Gaussian random variable.

(a) Show that for $\sigma > 0$ and $m \in \mathbb{R}$ the random variable $X = \sigma Z + m$ is also a Gaussian random variable with mean m and variance σ^2.

(b) Show that the moment generating function of a Gaussian random variable X with mean m and variance σ^2 is given by

$$\phi(t) = \mathbb{E}[e^t X] = e^{tm+t^2\sigma^2/2}.$$