MTH 383/683: Homework #3

Due Date: September 22, 2023

1 Problems for Everyone

1. **Sum of Exponential is Gamma**. Let $X_1, \ldots, X_n$ be n independent and identically distributed exponential random variables with parameter $\lambda > 0$ and let $Z = X_1 + \ldots + X_n$. Recall from the last homework that the moment generating function of each X_i is given by

   $$\phi(t) = \mathbb{E}[e^{tX_i}] = \frac{\lambda}{\lambda - t},$$

   for $t < \lambda$.

   (a) Find the moment generating function of Z.

   (b) A random variable Y is said to have a gamma distribution with parameter $\lambda > 0$ if it has the following probability density

   $$f(y) = \begin{cases} \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases},$$

   where $n \in \mathbb{N}$. Find the moment generating function Y and use this result to prove that the sum of n independent and identically distributed exponential random variables is a gamma distribution.

2. **Sum of Gaussian is Gaussian** Let X_1, X_2 be two independent but not necessarily identically distributed Gaussian random variables. Recall from last homework that the moment generating function of a Gaussian random variable with mean μ and standard deviation σ is given by

   $$\phi(t) = \mathbb{E}[e^{tX}] = e^{t\mu + t^2\sigma^2/2}.$$

   By computing its moment generating function, prove that $Z = X_1 + X_2$ is also a Gaussian random variable.

3. **Calculations with Joint Density** Let $\vec{X} = (X, Y)$ be a random vector with joint density

   $$f(x, y) = \begin{cases} 6x^2 y & 0 \leq x \leq y \text{ and } x + y \leq 2 \\ 0 & \text{elsewhere} \end{cases}.$$

   (a) Find the probability density functions of X and Y.

   (b) Are X and Y independent random variables?

4. **Example of Uncorrelated Random Variables that are not Independent** Let X be a standard Gaussian. Show that $\text{Cov}(X, X^2) = 0$, yet X and X^2 are not independent.

5. **The Covariance Matrix of a Random Vector is Always Positive Definite**. Let C be the covariance matrix of the random vector $X = (X_1, \dots X_n)$. Show that C is always positive semidefinite, i.e.,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i a_j C_{ij} \geq 0,$$

for any $a_1, \dots a_n \in \mathbb{R}^n$. **Hint:** Write the left side as the variance of some random variable.

6. **A Linear Transformation of a Gaussian Vector is also Gaussian**. Let $X = (X_1, \dots X_n)$ be an n-dimensional Gaussian vector and M a $m \times n$ matrix.

   (a) Show that $Y = MX$ is also a Gaussian vector.

   (b) If the covariance matrix of X is C, write the covariance matrix of Y in terms of M and C.