MTH 383/683: Homework #4

Due Date: Octoberr 06, 2023

1 Problems for Everyone

1. **Fractional Brownian Motion** Fractional Brownian motion B_t^H, with index $H \in (0,1)$, is a Gaussian process with mean 0 and covariance

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $\Delta t > 0$, let $\Delta B_t^H = B_{t+\Delta t}^H - B_t^H$.

   (a) For $t > 0$, calculate $\mathrm{Var}[B_t^H]$ and write down the probability density of B_t^H.

   (b) Compute $\mathbb{E}[\Delta B_t^H]$ and $\mathrm{Var}[\Delta B_t^H]$.

   (c) Compute $\mathrm{Cov}(\Delta B_t^H, \Delta B_s^H)$ if $s \le t$.

   (d) For fixed s and t, determine if the random variables ΔB_t^H and ΔB_s^H are independent.

2. **Brownian Bridge** Let B_t be a standard Brownian motion. The Brownian bridge is the stochastic process Z_t defined by

$$Z_t = B_t - tB_1$$

for $t \in [0,1]$. For $\Delta t > 0$, let $\Delta Z_t = Z_{t+\Delta t} - Z_t$.

   (a) Compute the exact values of Z_0 and Z_1.

   (b) Compute $\mathrm{Cov}(Z_t, Z_s)$ assuming $t > s$.

   (c) Compute $\mathrm{Var}(Z_t)$ and $\mathbb{E}[Z_t]$.

   (d) Compute $\mathbb{E}[\Delta Z_t]$ and $\mathrm{Var}[\Delta Z_t]$.

   (e) Compute $\mathrm{Cov}(\Delta Z_t, \Delta Z_s)$ if $s \le t$.

   (f) For fixed s and t, determine if the random variables ΔZ_t and ΔZ_s are independent.

3. **Shifted Brownian Bridge** Let B_t be a standard Brownian motion. For $t, s \in [0,1]$, construct a Gaussian process S_t that satisfies the following properties:

   - $\mathrm{Cov}(S_t, S_s) = s(1-t)$ if $s \le t$,
   - $S_0 = x_0$,
   - $S_1 = x_1$,

   where $x_0, x_1 \in \mathbb{R}$ are arbitrary.