MTH 383/683: Homework #6

Due Date: October 27, 2023

1 Problems for Everyone

1. **Conditional Expectation of Continuous Random Variables** Let X, Y be two random variables with joint density $f(x, y)$ on $\mathbb{R}^2$ and assume $f(x, y) > 0$ for all $x, y \in \mathbb{R}$. Show that the conditional expectation $\mathbb{E}[Y|X]$ equals $h(X)$ where h is the function

$$h(x) = \frac{\int_{-\infty}^{\infty} y f(x, y) dy}{\int_{-\infty}^{\infty} f(x, y)} dy.$$

   Hint: To prove this you need to show both properties of conditional expectation.

2. **Exercises on σ-fields** The Borel sets of $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra on $\mathbb{R}$ containing intervals of the form $(a, b]$. That is, $\mathcal{B}(\mathbb{R})$ contains all possible unions and intersections of intervals of the form $(a, b]$.

   (a) Show that all singletons $\{b\}$ are in $\mathcal{B}(\mathbb{R})$ by writing $\{b\}$ as the infinite intersection of intervals of the form $(b - 1/n, b + 1/n]$.

   (b) Prove that all open intervals (a, b) and closed intervals $[a, b]$ are in $\mathcal{B}(\mathbb{R})$.

3. **Another Look at Conditional Expectation for Gaussians** Let (X, Y) be a Gaussian vector with mean 0 and covariance matrix

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

   where $\rho \in (-1, 1)$.

   (a) Use Equation (4.5) in the text to show that $\mathbb{E}[Y|X] = \rho X$.

   (b) Write down the joint PDF $f(x, y)$ of (X, Y).

   (c) Show that

$$\int_{-\infty}^{\infty} y f(x, y) dy = \rho x \text{ and } \int_{-\infty}^{\infty} f(x, y) dy = 1.$$

   (d) Use problem #1 on this homework to show that $\mathbb{E}[Y|X] = \rho X$.

4. **Gaussian Conditioning** Consider the Gaussian vector (X_1, X_2, X_3) with mean 0 and covariance matrix

$$C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Prove that X_3 is independent of X_2 and X_1.

   (b) Compute $\mathbb{E}[X_2|X_1]$.

   (c) Write X_2 as a linear combination of X_1 and a random variable independent of X_1.

   (d) Compute $\mathbb{E}[e^{aX_2}|X_1]$ for any $a \in \mathbb{R}$.

5. Let B_t be a standard Brownian motion. Verify that $M_t = B_t^2 - t$ is a martingale for the Brownian filtration.