MTH 383/683: Homework #7

Due Date: November 10, 2023

1 Problems for Everyone

1. **Another Brownian Martingale:** Let B_t be standard Brownian motion. Consider for $a, b > 0$ the stopping time
$$\tau = \min_{t \geq 0}\{t : B_t \geq a \text{ or } B_t \leq -b\}.$$
   (a) Show that $M_t = tB_t - \frac{1}{3}B_t^3$ is a martingale for the Brownian filtration.
   (b) Use (a) to show that
$$\mathbb{E}[\tau B_\tau] = \frac{ab}{3}(a-b).$$

2. **Martingale Transform:** Let B_t be a standard Brownian motion on the interval $[0, 1]$ and let I_t be the stochastic process defined by
$$I_t = \begin{cases} 10B_t & t \in [0, 1/3] \\ 10B_{1/3} + 5(B_t - B_{1/3}) & t \in [1/3, 2/3] \\ 10B_{1/3} + 5(B_{2/3} - B_{1/3}) + 2(B_t - B_{2/3}) & t \in [2/3, 1] \end{cases}.$$
   (a) Show that I_t is martingale with respect to $\sigma(B_t)$.
   (b) Compute $\mathbb{E}[I_t^2]$.

3. **Ito Integral of Simple Process:** Let B_t be a standard Brownian motion and let I_t be the stochastic process defined by
$$I_t = \begin{cases} 0 & \text{if } s \in [0, 1/3] \\ B_{1/3}(B_t - B_{1/3}) & \text{if } s \in [1/3, 2/3] \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_s - B_{2/3}) & \text{if } s \in [2/3, 1] \end{cases}.$$
   (a) Show that I_t is a martingale.
   (b) Compute $\mathbb{E}[I_t^2]$.

4. **Increments of Martingales are not Correlated:** Let M_t be a martingale for the filtration $\mathcal{F}_t$. Use the properties of conditional expectation to show that for $t_1 < t_2 < t_3 < t_4$, we have
$$\mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t-3})] = 0.$$

5. **Not Everything is a Martingale** Show that a Gaussian process Y_t with the following covariance
$$C(Y_s, Y_t) = \frac{e^{-2(t-s)}}{2}(1 - e^{-2s})$$
is not a martingale.