MTH 383/683: Homework #8

Due Date: November 20, 2023

1 Problems for Everyone

1. **Practice on Ito Integrals:** Consider the four processes:

$$X_t = \int_0^t (1-s)dB_s, \qquad Y_t = \int_0^t (1+s)dB_s, \qquad Z_t = \int_0^t \sin(s)dB_s, \qquad W_t = \int_0^t \cos(s)dB_s.$$

   (a) Explain why each of these processes are Gaussian.

   (b) Find the mean and covariance for each of these processes.

   (c) Determine the probability densities for each of these processes.

   (d) For which time, if any, do we have that X_t and Y_t are uncorrelated? Are X_t and Y_t independent at these times?

   (e) Determine the covariance matrix for the Gaussian random variable $(Z_{\pi/2}, Z_\pi)$.

   (f) Write down the double integral for the probability $P(Z_{\pi/2} > 1, Z_\pi > 1)$.

   (g) Determine for which times the processes Z_t and W_t are indpendent.

2. **Integration By Parts for some Ito Integrals:** Let g be a smooth function and B_t a standard Brownian motion.

   (a) Use Ito's formula to prove that for any $t \geq 0$

$$\int_0^t g(s)dB_s = g(t)B_t - \int_0^t B_s g'(s)ds.$$

   (b) Show that the process given by

$$X_t = t^2 B_t - 2\int_0^t sB_s ds$$

   is Gaussian. Find its mean and covariance.

3. **Some Practice with Ito's Formula:** Show that:

   (a) $\displaystyle\int_0^t B_s^3 dB_s = \frac{1}{4}B_t^4 - \frac{3}{2}\int_0^t B_s^2 ds$

   (b) $\displaystyle\int_0^t B_s^4 dB_s = \frac{1}{5}B_t^5 - 2\int_0^t B_s^3 ds$

   (c) $\displaystyle\int_0^t B_s^{n-1} dB_s = \frac{1}{n}B_t^n - \frac{n-1}{2}\int_0^t B_s^{n-2} ds$, where $n \in \mathbb{N}$ and $n > 2$.

4. **Some Practice with Ito's Formula and Compensators:** Let B_t be a standard Brownian motion and consider the three processes:

$$X_t = \int_0^t \cos(s) dB_s, \qquad Y_t = B_t^4, \qquad Z_t = (B_t + t)\exp(-B_t - t/2).$$

   (a) Determine if each of these processes is a martingale, if not find a compensator for it.

   (b) Find the mean, the variance, and the covariance of each of these processes.

   (c) Determine if each of these processes are Gaussian.

5. **Gaussian Moments Using Ito:** Let B_t be a Brownian motion. Use Ito's formula to show that for $k \in \mathbb{N}$

$$\mathbb{E}[B_t^k] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_s^{k-2}]ds.$$

   Conclude from this that $\mathbb{E}[B_t^4] = 3t^2$ and $\mathbb{E}[B_t^6] = 15t^3$.