MTH 383/683: Homework #9

Due Date: December 01, 2023

1 Problems for Everyone

1. **Exercise on Ito's Formula:** Consider the process

$$X_t = \exp(tB_t).$$

   (a) Explain briefly why $Z_t = tB_t$ is a Gaussian random variable and find its mean and variance.

   (b) Find the mean and variance of X_t.

   (c) Use Ito's formula to write Z_t in terms of an Ito integral and a Riemann integral.

   (d) Find a compensator C_t so that $X_t - C_t$ is a martingale.

   (e) Show that the covariance between B_t and $\int_0^t e^{sB_s} dB_s$ at time t is

$$\mathrm{Cov}\left(B_t, \int_0^t e^{sB_s} dB_s\right) = \int_0^t e^{s^3/2} ds.$$

2. **Martingales and Ito's Formula:** Prove that

$$Y_t = e^{t/2} \cos(B_t)$$

   is a martingale.

3. **Random Time Blowup:** Consider the following stochastic differential equation:

$$\begin{aligned} dX &= -\frac{1}{2} e^{-2X} dt + e^{-X} dB \\ X(0) &= x_0. \end{aligned}$$

   (a) Use the substitution $X = u(B)$ to solve this stochastic differential equation.

   (b) Show that the solution diverges at a finite but random time.

4. **Solving an SDE:** Solve the following stochastic differential equations

   (a) $dX = -X dt + e^{-t} dB$, $X(0) = 0$

   (b) $dX = -X/(1+t) dt + 1/(1+t) dB$, $X(0) = 1$

   (c) $dX = -X/(1-t) dt + dB$, $X(0) = 0$

   (d) $dX = XB dt + dB$, $X(0) = x_0$.

5. **Brownian Motion on the Unit Circle:** Consider the following system of stochastic differential equations

$$\begin{cases} dX &= -\frac{1}{2}X\,dt - Y\,dB \\ dY &= -\frac{1}{2}Y\,dt + X\,dB \end{cases}.$$

(a) Show that for any solution to this stochastic differential equation, $X^2 + Y^2$ is constant in time.

(b) Show that $X = (\cos(B), \sin(B))$ solves this system.