Lecture 11: Martingales

Definition - A filtration $(\mathcal{F}_t, t \geq 0)$ is an increasing sequence of σ-algebras:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \ , \ s \leq t$$

*Usually $\mathcal{F}_0 = \{\phi, \Omega\}$.

Example:

If X_t is a stochastic process, the natural filtration of a given process is

$$\mathcal{F}_t = \sigma(X_s, s \leq t).$$

Definition - A process M_t is a Martingale for the filtration $\mathcal{F}_t$ if the following hold:

1. The process is adapted: M_t is $\mathcal{F}_t$ measurable.
2. $\mathbb{E}[|M_t|] < \infty$
3. Martingale Property:

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s$$

for all $s \leq t$.

Example:

If M_t is a Martingale then $\mathbb{E}[M_t] = \mathbb{E}[M_0]$.

proof:

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \text{ for all } s \leq t$$

$$\Rightarrow \mathbb{E}[M_t | \mathcal{F}_0] = M_0$$

$$\Rightarrow \mathbb{E}[\mathbb{E}[M_t | \mathcal{F}_0]] = \mathbb{E}[M_0]$$

$$\Rightarrow \mathbb{E}[M_t] = \mathbb{E}[M_0]$$

Example:

Let B_t be standard Brownian motion and $\mathcal{F}_t = \sigma(B_s, s \le t)$.

$$\Rightarrow \mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s \mid \mathcal{F}_s]$$
$$= \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] + \mathbb{E}[B_s \mid \mathcal{F}_s]$$
$$= \mathbb{E}[B_t - B_s] + B_s$$
$$= B_s$$

$\rightarrow$ Brownian motion is a martingale.

Example: $\longrightarrow$ geometric Brownian motion

$S_t = S_0 \exp(\sigma B_t + \nu t)$, $\mathcal{F}_t = \sigma(B_s, s \le t)$.

$$\Rightarrow \mathbb{E}[S_t \mid \mathcal{F}_s] = \mathbb{E}[S_0 e^{\sigma B_t + \nu t} \mid \mathcal{F}_s]$$
$$= S_0 e^{\nu t} \mathbb{E}[e^{\sigma B_t} \mid \mathcal{F}_s]$$
$$= S_0 e^{\nu t} \mathbb{E}[e^{\sigma(B_t - B_s + B_s)} \mid \mathcal{F}_s]$$
$$= S_0 e^{\nu t} (\mathbb{E}[e^{\sigma(B_t - B_s)} e^{\sigma B_s} \mid \mathcal{F}_s])$$
$$= S_0 e^{\nu t} e^{\sigma B_s} \mathbb{E}[e^{\sigma(B_t - B_s)}]$$
$$= S_0 e^{\nu t} e^{\sigma B_s} e^{2\sigma^2 t}$$
$$= S_0 e^{\nu t + \frac{\sigma^2(t-s)}{2} + \sigma B_s}$$
$$= S_0 e^{\nu(t-s) + \sigma^2/2 (t-s)} e^{\nu s + \sigma B_s} = e^{\nu(t-s) + \sigma^2/2(t-s)} S_s$$

$\Rightarrow \mathbb{E}[S_t \mid \mathcal{F}_s] \ge S_s \Rightarrow S_t$ is a submartingale.

If $\nu = -\sigma^2/2$ then S_t is a martingale.