Lecture 12: Stopping Times

Definition ~ A random variable is said to be a stopping time for the filtration $\mathcal{F}_t$ if

$$\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t$$

* τ is a stopping time if the events $\{\tau \le t\}$ can be determined to occur or not based on information at time t.

Examples:

1. First Passage Time: Let X_t be a process with its natural filtration. The first passage time to $a \in \mathbb{R}$ is

$$\tau(\omega) = \min\{s \ge 0 : X_s(\omega) \ge a\}.$$

2. Hitting Time: Let X_t be a process with it natural filtration and $B \subset \mathbb{R}$. The hitting time to B is

$$\tau(\omega) = \min\{s \ge 0 : X_s(\omega) \in B\}.$$

3. Minimum of two times: If τ and τ' are two stopping times for the same filtration $\mathcal{F}_t$ then so is

$$\tau'' = \min\{\tau, \tau'\}.$$

* This works since

$$\{\omega : \min\{\tau', \tau\} \le t\} = \{\omega : \tau \le t\} \cup \{\omega : \tau' \le t\}.$$

Example:

If M_t is a martingale for the filtration $\mathcal{F}_t$ and τ is a stopping time for the same filtration, then the stopped process defined by

$$M_{\min\{t,\tau\}} = \begin{cases} M_t, & t \leq \tau \\ M_\tau, & t \geq \tau \end{cases}$$

is a martingale.

Theorem (Doob's Optional Stopping Theorem): If M_t is a continuous martingale for the filtration $\mathcal{F}_t$ and τ is a stopping time such that $\tau < \infty$ and the stopped process is bounded then

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

~~proof~~:

Since $M_{\min\{t,\tau\}}$ is a martingale we have that

$$\mathbb{E}[M_{\min\{t,\tau\}}] = \mathbb{E}[M_0]$$

Now, since $\tau < \infty$ we have

$$\lim_{t\to\infty} M_{\min\{t,\tau\}} = M_\tau$$

Therefore, (by the dominated convergence theorem)

$$\mathbb{E}[M_\tau] = \mathbb{E}[\lim_{t\to\infty} M_{\min\{t,\tau\}}] = \lim_{t\to\infty} \mathbb{E}[M_{\min\{t,\tau\}}] = \lim_{t\to\infty} \mathbb{E}[M_0] = \mathbb{E}[M_0]$$

Example:

What is the probability B_t reaches a before $-b$?

Let $\tau(\omega) = \min\{t : B_t(\omega) \geq a \text{ or } B_t(\omega) \leq -b\}$.

- We first show that $\tau(\omega) < \infty$. Let

$$E_n = \{|B_n - B_{n-1}| > a+b\},\ n \geq 1$$

There $P(E_n) = P(E_1) = p$. Since E_n are independendent we have

$$P(E_1^c \cap \cdots \cap E_n^c) = (1-p)^n$$

$$\Rightarrow P\left(\bigcap_{i=1}^{\infty} E_i^c\right) = \lim_{n\to\infty} (1-p)^n = 0.$$

Therefore, E_n occurs for some n and so the B_t crosses a or $-b$ eventually.

- Now, we have by Doob's optional stopping time

$$0 = \mathbb{E}[B_\tau] = a P(B_\tau = a) - b P(B_t = -b)$$

$$= a P(B_t = a) - b(1 - P(B_t = a))$$

$$\Rightarrow P(B_\tau = a) = \frac{b}{a+b}, \quad P(B_\tau = -b) = \frac{a}{a+b}$$

Example:

What is the expected waiting time for the path to reach a or b? Let $M_t = B_t^2 - t$. Therefore,

$$\mathbb{E}[M_\tau] = M_0 = 0$$

$$\Rightarrow \mathbb{E}[B_\tau^2 - \tau] = 0$$

$$\Rightarrow \mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$$

$$\Rightarrow a^2 \cdot \frac{b}{a+b} + \frac{b^2 a}{a+b} = \mathbb{E}[\tau]$$

$$\Rightarrow \mathbb{E}[\tau] = ab.$$

Example:

Let $\tau_a = \min\{t \geq 0 : B_t \geq a\}$. Let $\tau = \min\{\tau_a, \tau_{-b}\}$ with $\tau_{-b} = \min\{t \geq 0, B_t \leq -b\}$. Therefore,

$$\lim_{b\to\infty} P(\tau_a < \tau_{-b}) = P(\tau_a < \infty).$$

On the other hand,

$$\lim_{b\to\infty} P(\tau_a < \tau_{-b}) = \lim_{b\to\infty} P(B_\tau = a) = \lim_{b\to\infty} \frac{b}{a+b} = 1.$$

$$\Rightarrow P(\tau_a < \infty) = 1.$$

We also know that

$$\mathbb{E}[\tau] = ab$$

$$\Rightarrow \lim_{b\to\infty} \mathbb{E}[\tau] = \lim_{b\to\infty} \mathbb{E}[\min\{\tau_a, \tau_b\}] = \mathbb{E}[\lim_{b\to\infty} \min\{\tau_a, \tau_b\}] = \mathbb{E}[\tau_a]$$

$$\Rightarrow \mathbb{E}[\tau_a] = \infty.$$