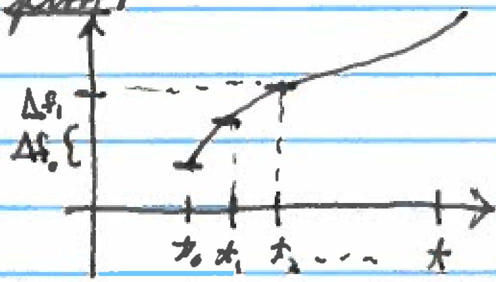
Lecture 14: Itô's Formula

Fundamental Theorem of Calculus

$$f(t) - f(t_0) = \int_{t_0}^{t} f'(s)\,ds \Rightarrow df = f'(t)\,dt$$

~~proof~~:



$$\Delta f_i = f_{i+1} - f_i$$

$$f(t) - f(t_0) = f(t_{i+1}) - f(t_i) + f(t_i) - f(t_{i-1}) + \ldots + f(t_1) - f(t_0)$$
$$= \sum_{i=0}^{n-1} \Delta f_i$$

However,

$$\Delta f_i = f'(t_i)\Delta x + \tfrac{1}{2} f''(t_i)\Delta x^2 + \ldots$$

$$\Rightarrow \sum_{i=0}^{n-1} \Delta f_i = \sum_{i=0}^{n-1} f'(t_i)\Delta x + \frac{\Delta x}{2}\sum_{i=0}^{n-1} f''(t_i)\Delta x + \ldots$$

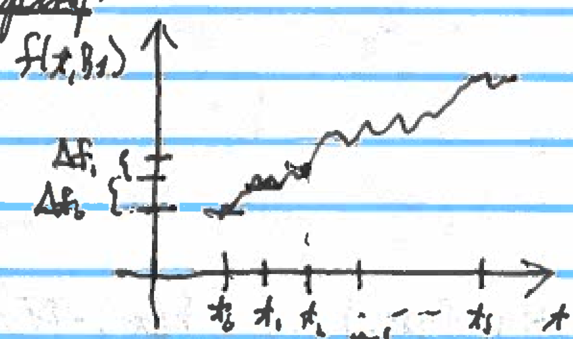
$$\rightarrow \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta f_i = \int_{t_0}^{t_f} f'(t_i)\,dt + 0$$

Ito's Formula

$$f(t, B_t) - f(t_0, B_0) = \int_0^t \frac{\partial f}{\partial B}\, dB_s + \int_0^t \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) dt$$

$$\Rightarrow df = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) dt + \frac{\partial f}{\partial B}\, dB_s.$$

~~proof~~:



$$f(t_f) - f(t_0) = \sum_{i=0}^{n-1} \Delta f_i$$

$$\Delta f_i = \frac{\partial f}{\partial t} \Delta t + \frac{\partial f}{\partial B} \Delta B + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2} \Delta t^2 + 2 \frac{\partial^2 f}{\partial t \partial B} \Delta t \Delta B + \frac{\partial^2 f}{\partial B^2} \Delta B^2 \right) + \dots$$

$$\Rightarrow f(t_f) - f(t_0) = \sum_{i=0}^{n-1} (f_t \Delta t + f_B \Delta B) + \Delta t \left(\sum_{i=0}^{n-1} \tfrac{1}{2} f_{tt} \Delta t + f_{tB} \Delta B \right) + \frac{1}{2} \sum_{i=0}^{n-1} f_{BB} \Delta B^2$$

Since $\mathbb{E}[\Delta B^2] = \Delta t$, in the limit we obtain

$$f(t_f) - f(t_0) = \lim_{n \to \infty} \sum_{i=0}^{n-1} \Delta f_i = \int_{t_0}^{t_f} \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) dt + \int_{t_0}^{t_f} \frac{\partial f}{\partial B}\, dB.$$

Corollary:

The process

$$f(t, B_t) - \underbrace{\int_0^t \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) ds}_{\text{compensator.}}$$

is a martingale.

proof:

By Ito's formula

$$f(t, B_t) - \int_0^t \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) ds = \int_0^t \frac{\partial f}{\partial B} dB$$

which is a martingale.

Example:

$$f(B_t) = B_t^3$$

$$\Rightarrow f(B_t) - f(B_0) = \int_0^t f'(B_t)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, ds$$

$$\Rightarrow B_t^3 = \int_0^t 3 B_s^2\, dB_s + \frac{1}{2} \int_0^t 6 B_s\, ds$$

Therefore,

$$B_t^3 - 3 \int_0^t B_s\, ds$$

is a martingale.

Example:

$$f(t, B_t) = \exp(\sigma B_t + \nu t)$$

$$\Rightarrow df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} dB^2$$

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} dt^2$$

$$\Rightarrow df = \left(\nu + \frac{\sigma^2}{2} \right) \exp(\sigma B_t + \nu t)\, dt + \frac{\sigma}{2} \exp(\sigma B_t + \nu t)\, dB.$$

$$\Rightarrow \boxed{df = \left(\nu + \frac{\sigma^2}{2} \right) f\, dt + \frac{\sigma}{2} f\, dB.}$$