Lecture 15: Stochastic Differential Equations.

Example:

S is the random price of a stock. Assume the change in the stock price in time Δt is given by

$$\Delta S = \underbrace{\mu S \Delta t}_{\text{growth from interest}} + \underbrace{\sigma S \Delta B}_{\text{randomness in market}}$$

Assuming $\Delta t \to 0$ we obtain

$$\boxed{\begin{aligned} &dS = \mu S\,dt + \sigma S\,dB \\ &S(0) = S_0 \to \text{starting stock price} \end{aligned}}$$

This equation is linear with constant coefficients so we make a guess

$$S = S_0 e^{\lambda_1 t + \lambda_2 B}$$

$$\Rightarrow dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial B}dB + \frac{1}{2}\frac{\partial^2 S}{\partial B^2}dt$$

$$= S_0 e^{\lambda_1 t + \lambda_2 B}\left((\lambda_1 + \tfrac{1}{2}\lambda_2^2)dt + \lambda_2 dB\right)$$

Since $dS = \mu S\,dt + \sigma S\,dB$ we have that

$$S_0 e^{\lambda_1 t + \lambda_2 B}\left((\lambda_1 + \tfrac{1}{2}\lambda_2^2)dt + \lambda_2 dB\right) = S_0 e^{\lambda_1 t + \lambda_2 B}(\mu\,dt + \sigma\,dB)$$

$$\Rightarrow \lambda_2 = \sigma$$
$$\lambda_1 + \tfrac{1}{2}\sigma^2 = \mu$$

$$\Rightarrow \lambda_2 = \sigma$$
$$\lambda_1 = \mu - \tfrac{1}{2}\sigma^2$$

Therefore,

$$\boxed{S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B}}$$

→ If $\mu > \frac{1}{2}\sigma^2$ expect growth

→ If $\mu < \frac{1}{2}\sigma^2$ expect decay.

Since $(\nu-\frac{1}{2}\sigma^2)t+\sigma B_t$ is Gaussian it follows that

$$
\begin{aligned}
\mathbb{E}[S_t] &= \mathbb{E}[e^{(\nu-\frac{1}{2}\sigma^2)t+\sigma B_t}] \\
&= e^{(\nu-\frac{1}{2}\sigma^2)t}\,\mathbb{E}[e^{\sigma B_t}] \\
&= e^{(\nu-\frac{1}{2}\sigma^2)t}\,e^{\frac{1}{2}\sigma^2 t} \\
&= e^{\nu t}
\end{aligned}
$$

*Note, $e^{\nu t}$ solves the deterministic ODE

$$\frac{dS}{dt} = \nu S, \quad S(0) = S_0. \longrightarrow$$ Expected value tracks deterministic dynamics.

Furthermore,

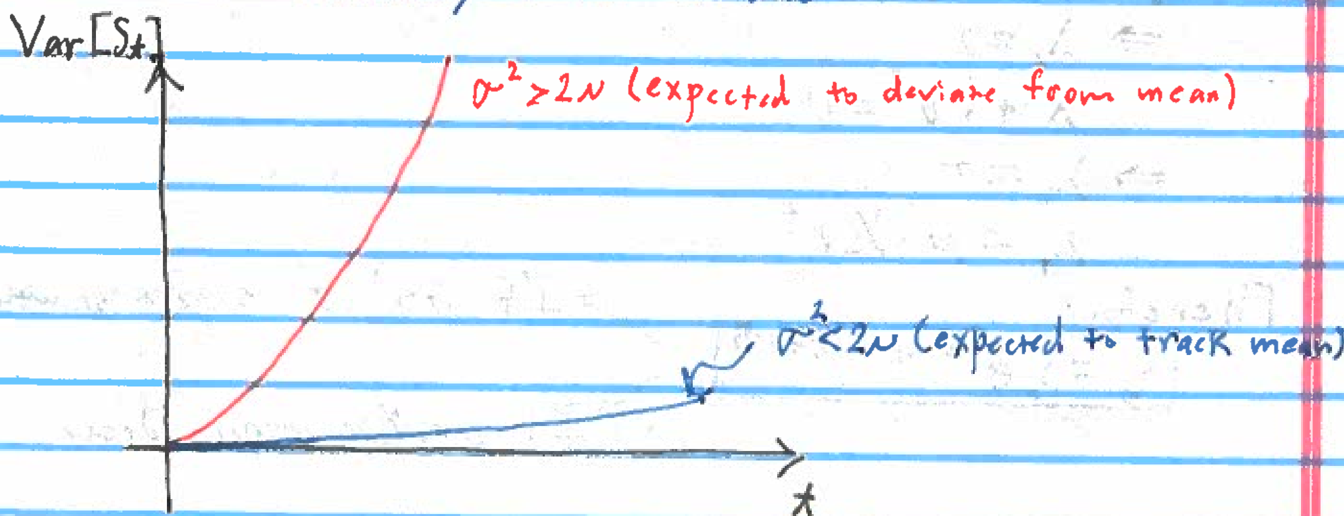
$$
\begin{aligned}
\mathbb{E}[S_t^2] &= \mathbb{E}[e^{(2\nu-\sigma^2)t+2\sigma B}] \\
&= e^{(2\nu-\sigma^2)t}\,\mathbb{E}[e^{2\sigma B}] \\
&= e^{(2\nu-\sigma^2)t}\,e^{2\sigma^2 t} \\
&= e^{2\nu t}e^{\sigma^2 t}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
Var[S_t] &= \mathbb{E}[S_t^2]-\mathbb{E}[S_t]^2 \\
&= e^{2\nu t}e^{\sigma^2 t} - e^{2\nu t} \\
&= e^{2\nu t}(e^{\sigma^2 t}-1)
\end{aligned}
$$

essentially zero for $t < 1/\sigma^2$

essentially 1 for $t < 1/2\nu$



Since $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B}$ it follows that

$S_t \sim e^{\sigma B}$ for large t. (No Better than gambling).

Example (ODE):

Solve

$$\frac{dx}{dt} + \frac{x}{1+t} = e^t$$

$$x(0) = 0$$

$$\Rightarrow e^{g(t)}\frac{dx}{dt} + e^{g(t)}\frac{x}{1+t} = e^{g(t)}e^t$$

If

$$\frac{d}{dt}\left(e^{g(t)}x\right) = e^{g(t)}\frac{dx}{dt} + e^{g(t)}\frac{x}{1+t}$$

then $g'(t) = \frac{1}{1+t} \Rightarrow g(t) = \ln(1+t)$. Consequently,

$$\frac{d}{dt}\left(e^{\ln(1+t)}x\right) = (1+t)e^t$$

$$\Rightarrow d((1+t)x) = (1+t)e^t dt$$

$$\Rightarrow (1+t)x = \int_0^t (1+s)e^s ds$$

$$\Rightarrow (1+t)x = te^t$$

$$\Rightarrow \boxed{x(t) = \frac{te^t}{1+t}.}$$

Example:

Solve

$$dX = -tX\,dt + e^{-t}\,dB$$
$$X(0) = X_0$$
$$\Rightarrow dX + tX\,dt = e^{-t}\,dB$$
$$\Rightarrow e^{g(t)}dX + tXe^{g(t)}dt = e^{g(t)}e^{-t}dB$$

If

$$d(e^{g(t)}X) = e^{g(t)}dX + tXe^{g(t)}dt$$

then

$$g'(t) = t$$
$$\Rightarrow g(t) = \frac{t^2}{2}$$
$$\Rightarrow d(e^{t^2/2}X) = e^{t^2/2 + t}dB$$
$$\Rightarrow e^{t^2/2}X - X_0 = \int_0^t e^{s^2/2+s}dB$$

$$\boxed{X(t) = e^{-t^2/2}X_0 + e^{-t^2/2}\int_0^t e^{s^2/2+s}dB_s}$$

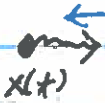
$$\Rightarrow \mathbb{E}[X(t)] = X_0 e^{-t^2/2}$$
$$\Rightarrow \mathbb{E}[X^2(t)] = \mathbb{E}\left[e^{-t^2}X_0^2 + 2e^{-t^2}X_0\int_0^t e^{s^2/2+s}dB_s + e^{-t^2}\left(\int_0^t e^{s^2/2+s}dB_s\right)^2\right]$$
$$= e^{-t^2}X_0^2 + e^{-t^2}\int_0^t (e^{s^2+2s})\,ds$$

$$\Rightarrow \boxed{\mathrm{Var}(X(t)) = e^{-t^2}\int_0^t e^{2s}e^{s^2}ds}$$

Example:

Using physics to model Brownian motion in fluid:

← fluid resistance pushes particle back.



Newton's Law

$$m\frac{d^2x}{dt^2} = \text{Forces}$$

Let $v = \frac{dx}{dt}$ (velocity)

$$\Rightarrow \frac{dx}{dt} = v$$

$$m\frac{dv}{dt} = \text{Forces} = \text{"drag"} + \text{"random fluctuations"}$$

coefficient of friction

$$\text{drag} = -\gamma v,$$

$$\text{fluctuations} = \text{white noise} = \lambda\frac{dB}{dt}$$

$$\Rightarrow \boxed{\begin{aligned} dx &= v\,dt \\ m\,dv &= -\gamma v\,dt + \lambda\,dB \end{aligned}}$$

→ Ornstein-Uhlenbeck process / Langevin Equation

Case 1:

If $m \ll 1$ then

$$\gamma v\,dt = \lambda\,dB$$

$$\gamma\,dx = \lambda\,dB$$

$$\Rightarrow dx = \sigma\,dB \quad (\sigma = \lambda/\gamma)$$

(Recover Classic Brownian motion)

Case 2:

If $m \sim \gamma \sim \lambda$ then

$$dx = v\,dt$$
$$dv = -\nu v\,dt + \sigma\,dB \quad (\nu = \gamma/m,\ \sigma = \lambda/m)$$
$$x(0) = 0$$
$$v(0) = 0$$

$$\Rightarrow dv + \nu v\,dt = \sigma\,dB$$
$$\Rightarrow d(e^{\nu t} v) = \sigma e^{\nu t} dB$$
$$\Rightarrow e^{\nu t} v = \sigma \int_0^t e^{\nu s} dB_s$$
$$\Rightarrow \boxed{v = \sigma e^{-\nu t} \int_0^t e^{\nu s} dB_s}$$ ← Gaussian process with mean zero.

$$\Rightarrow dx = \sigma e^{-\nu t} \int_0^t e^{\nu s} dB_s$$

$$\boxed{x(t) = \sigma \int_0^t \int_0^u e^{\nu(s-u)} dB_s\,du}$$ ← Gaussian process with mean zero.

$$\mathbb{E}[V_t V_{t'}] = \sigma^2 e^{-\nu t} e^{-\nu t'} \mathbb{E}\left[\int_0^t e^{\nu s} dB_s \int_0^{t'} e^{\nu s'} dB_{s'}'\right]$$
$$= \sigma^2 e^{-\nu(t+t')} \int_0^{\min\{t,t'\}} e^{2\nu s} ds$$
$$= \frac{\sigma^2 e^{-\nu(t+t')}}{2\nu} e^{2\nu \min\{t,t'\}}$$
$$= \frac{\sigma^2 e^{\nu|t-t'|}}{2\nu}$$

$$\Rightarrow \boxed{\mathrm{Cov}(V_t, V_{t'}) = \frac{\sigma^2}{2\nu} \exp(\nu|t-t'|)}$$