Lecture #3: Expectation

* The expected value of a random variable is simply the weighted average by the probability:

$$\Rightarrow \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x\,dF \quad \text{continuous r.v.}$$

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\{\omega\}) = \sum_{x} x\, P(X = x). \quad \text{discrete r.v.}$$

Example:

Let X be a random variable with an exponential distribution with parameter $\lambda > 0$. Compute $\mathbb{E}[X]$.

$$\mathbb{E}[X] = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx$$
$$= -x e^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$
$$= 1/\lambda.$$

Important Expectations

1. Variance: $\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$

$$= \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x)\,dx$$
$$= \int_{-\infty}^{\infty} (x^2 - 2x\,\mathbb{E}[X] + \mathbb{E}[X^2]) f(x)\,dx$$
$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\,\mathbb{E}[X] \int_{-\infty}^{\infty} x f(x)\,dx + \mathbb{E}[X^2] \int_{-\infty}^{\infty} f(x)\,dx$$
$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]\,\mathbb{E}[X] + \mathbb{E}[X]^2$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

2. Moments: The n-th moment is $\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx$.

3. Probability: $\mathbb{E}[\mathbb{1}_B] = \int_{-\infty}^{\infty} \mathbb{1}_B(x) f(x)\,dx$

$$= \int_B f(x)\,dx$$
$$= P(X \in B).$$

4. Moment Generating Function: Let $\lambda \in \mathbb{R}$. The moment generating function is

$$\phi(\lambda) = \mathbb{E}[e^{\lambda X}]$$
$$= \int_{-\infty}^{\infty} e^{\lambda x} f(x)\,dx$$
$$= \int_{-\infty}^{\infty} (1 + \lambda x + \frac{\lambda^2}{2} x^2 + \frac{\lambda^3}{6} x^3 + \dots) f(x)\,dx$$
$$\Rightarrow \phi^{(n)}(0) = \mathbb{E}[X^n].$$

Inequalities

1. Markhov's Inequality: Let X be a positive random variable on $(\Omega, \mathcal{F}, P)$. Then for any $a > 0$ we have

$$P(X > a) \le 1/a\, \mathbb{E}[X]$$

proof

$$\mathbb{E}[X] = \int_0^{\infty} x f(x)\,dx$$
$$= \int_0^a x f(x)\,dx + \int_a^{\infty} x f(x)\,dx$$
$$\ge \int_a^{\infty} x f(x)\,dx$$
$$\ge a \int_a^{\infty} f(x)\,dx$$
$$= a P(X \ge a).$$

2. Chebyshev's Inequality: For any random variable on $(\Omega, \mathcal{F}, P)$

$$P(|X| > a) \le \frac{1}{a^2} \mathbb{E}[X^2]$$

proof:

$$P(|X| > a) = P(X^2 > a^2) \le \frac{1}{a^2} \mathbb{E}[X^2]$$

$$\Rightarrow P(|X - \mathbb{E}[X]| > n\sigma) \le \frac{1}{n^2\sigma^2} \mathbb{E}[(X - \mathbb{E}[X])^2] = \frac{1}{n^2}.$$

3. Chernoff bound: $P(X > a) \le e^{-\lambda a}\, \mathbb{E}[e^{\lambda X}]$

proof:

$$P(X > a) = P(e^{\lambda X} > e^{\lambda a}) \le \frac{1}{e^{\lambda a}} \mathbb{E}[e^{\lambda X}].$$