Lecture 4: Random Vectors

On a probability space $(\Omega, \mathcal{F}, P)$ we can have n random variables $X_1, \dots, X_n$ such that $X_i : \Omega \to \mathbb{R}$. This is called a random vector and can also be written $\vec{X} : \Omega \to \mathbb{R}^n$.

The distribution of $\vec{X}$ is defined by

$$\mu_{\vec{X}}(A) = P(\vec{X} \in A) \text{ for } A \subseteq \mathbb{R}^n.$$

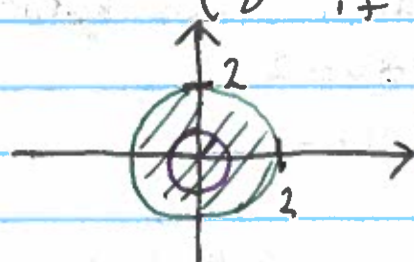
The joint density function (PDF) is defined by

$$P(\vec{X} \in A) = \int_A f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

Example:

$\vec{X} = (X, Y)$ where $\vec{X} : \Omega \to \mathbb{R}^2$ is the (x,y) coordinate of a point chosen uniformly in a circle of radius 2.

$$f(x,y) = \begin{cases} \frac{1}{4\pi} & \text{if } x^2+y^2 \le 4 \\ 0 & \text{if } x^2+y^2 > 1 \end{cases}$$

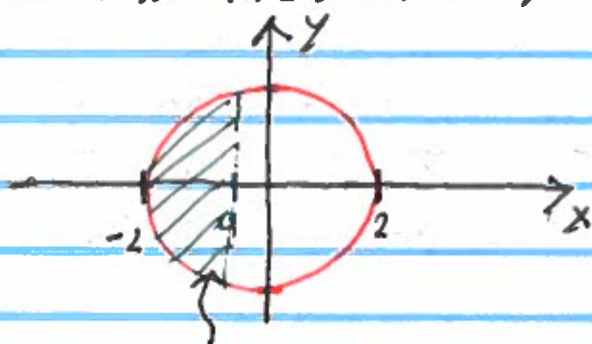


What is $P(\vec{X} \in \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\})$?

$$P(\vec{X} \in \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}) = \iint_{x^2+y^2 \le 1} \frac{1}{4\pi}\, dx\, dy = \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \frac{1}{2}.$$

X, Y are also random variables. What is the PDF of X?

$P(\mathbb{X} \leq a) = \int_{\{(x,y)\in\mathbb{R}^2 : x \leq a\}} f(x,y)\,dx\,dy = F_{\mathbb{X}}(x)$



integral over this region.

$$\Rightarrow P(\mathbb{X} \leq a) = \int_{-2}^{a}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{4\pi}\,dy\,dx$$

$$= \int_{-2}^{a} \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$

$$= F_{\mathbb{X}}(a)$$

$$\Rightarrow \frac{dF_{\mathbb{X}}}{da} = f(a) = \frac{1}{2\pi}\left(\sqrt{4-a^2}\right)$$

The PDF is therefore,

$$f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & -2 \leq x \leq 2 \\ 0, & |x| > 2 \end{cases}.$$

Similarly, the PDF for $\mathbb{Y}$ is given by

$$g(y) = \begin{cases} \frac{1}{2\pi}\sqrt{4-y^2}, & -2 \leq y \leq 2 \\ 0, & |y| > 2 \end{cases}.$$

The densities of $\mathbb{X}, \mathbb{Y}$ are called the <u>marginal densities.</u>

Random variables $\mathbb{X}_1, \dots, \mathbb{X}_n$ are independent if for all $A \in \mathcal{F}$:

$$P(\mathbb{X}_1 \in A_1, \dots, \mathbb{X}_n \in A_n) = P(\mathbb{X}_1 \in A_1) \cdots P(\mathbb{X}_n \in A_n)$$

$$\Rightarrow f(x_1, \dots, x_n) = f(x_1) \cdots f(x_n) \leftarrow \text{product of marginals.}$$

$$= P(\vec{\mathbb{X}} \in A_1 \cap \dots \cap A_n) = P(X_1 \in A_1 \text{ and } X_2 \in A_2 \text{ and} \dots X_n \in A_n).$$

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Example:

Let (X, Y) be independent, identically distributed with standard Gaussian distributions.

1. What is $P(X^2+Y^2<1)$?

$$P(X^2+Y^2\le 1) = \iint_{x^2+y^2\le 1} f(x,y)\,dx\,dy$$

I.I.D.

$$= \iint_{x^2+y^2\le 1} \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \frac{1}{2\pi} e^{-(x^2-y^2)/2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2\pi} e^{-r^2/2}\, r\,dr\,d\theta$$

density in polar coordinates.

$$= \int_0^{2\pi} -e^{-r^2/2}\Big|_0^1 d\theta$$

$$= \frac{1}{2\pi}\left(1 - e^{-1/2}\right).$$

2. Let $R = (X^2+Y^2)^{1/2}$. Then

$$\mathbb{E}[R] = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2} r^2\,dr$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2/2} r^2\,dr\,d\theta$$

$$= \frac{2\pi}{\sqrt{2\pi}} \cdot \frac{1}{2}$$

$$= \sqrt{\pi/2}.$$

3. What is the PDF of R?

$$P(R\le r) = \int_0^{2\pi}\int_0^{r} \frac{1}{2\pi} e^{-s^2/2}\, s\,dr\,d\theta$$

$$= \int_0^{r} e^{-s^2/2}\, s\,dr$$

$$\Rightarrow f(r) = r e^{-r^2/2}.$$

Definition - If (X,Y) is a random vector, then the covariance is given by

$$\text{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])]\mathbb{E}[(Y-\mathbb{E}[Y])]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

For a vector $X = (X_1, \ldots, X_n)$ the covariance matrix is given by

$$C_{ij} = \text{Cov}(X_i, X_j).$$

* Covariance is the expected change of one random variable with respect to another.

$\Rightarrow$ If $C_{ij} < 0$ then if X_i increases X_j decreases.