Lecture 5: Gaussian Vectors

Definition - An n-dimensional random vector $(X_1, \dots, X_n)$ is said to be Gaussian if any linear combination of $X_1, \dots, X_n$ is a Gaussian random variable:

$$a_1 X_1 + \dots + a_n X_n \text{ is Gaussian for any } a_1, \dots, a_n \in \mathbb{R}.$$

This is sometimes called jointly Gaussian.

1. Each X_i is Gaussian.
2. If M is an $m \times n$ matrix then MX is also Gaussian.
3. $\mathbb{E}[a_1 X_1 + \dots + a_n X_n] = a_1 \mu_1 + \dots + a_n \mu_n = \vec{a}^T \cdot \mu$
4. $\mathrm{Var}[(a_1 X_1 + \dots + a_n X_n)] = \mathbb{E}[(a_1(X_1 - \mu_1) + \dots + a_n(X_n - \mu_n))^2]$

$$= \mathbb{E}[(\sum_{i=1}^{n} a_i (X_i - \mu_i))^2]$$

$$= \mathbb{E}[\sum_{i=1}^{n} a_i (X_i - \mu_i) \sum_{j=1}^{n} a_j (X_j - \mu_j)]$$

$$= \mathbb{E}[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j (X_i X_j - \mu_i X_j - \mu_j X_i + \mu_i \mu_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j (\mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j])$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j C_{ij}$$

$$\Rightarrow \mathrm{Var}[(a_1 X_1 + \dots + a_n X_n] = a^T C a.$$

Definition - The joint MGF of $(X_1, \dots, X_n)$ is the function $\phi: \mathbb{R}^n \to \mathbb{R}$ defined by

$$\phi(\underline{a}) = \mathbb{E}[\exp(a_1 X_1 + \dots + a_n X_n)]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{a_1 x_1 + \dots a_n x_n} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

for $a = (a_1, \dots, a_n)$.

Theorem - A random vector is Gaussian if and only if the moment generating function of $\vec{X}$ is

$$\mathbb{E}[e^{a^T X}] = \exp(a^T \mu + \tfrac{1}{2} a^T C a).$$

proof:

$\vec{X}$ is Gaussian if and only if for all $\vec{a} \in \mathbb{R}^n$, $a^T \vec{X}$ is a Gaussian random variable. The mean and variance of $a^T X$ are

$$\mu^* = a^T \vec{\mu} \quad \text{and} \quad \sigma^2 = \vec{a}^T C \vec{a}$$

and thus has moment generating function

$$\phi(t) = \exp(t a^T \vec{x} + t^2 a^T C a / 2).$$

Proposition - Let $\vec{X} = (X_1, \dots, X_n)$ be a Gaussian vector with invertible covariance matrix C. Then the joint density of X is given by

$$f(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2} (\det C)^{1/2}} \exp\left(-\frac{1}{2} (\vec{x} - \mu)^T C^{-1} (\vec{x} - \mu) \right).$$

Example: - Consider the Gaussian vector (X_1, X_2) given by

$$X_1 = Z_1 + Z_2$$
$$X_2 = Z_1 - Z_2$$

where (Z_1, Z_2) are IID standard Gaussians.

$$\Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \mathrm{Cov}(X_1, X_2) &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\ &= \mathbb{E}[(Z_1 + Z_2)(Z_1 - Z_2)] \\ &= \mathbb{E}[Z_1^2 - Z_2^2] \\ &= \mathbb{E}[Z_1^2] - \mathbb{E}[Z_2^2] \\ &= 1 - 1 \\ &= 0 \end{aligned}$$