Lecture 6: Gaussian Processes

Stochastic Process

$$X = (X_t, t \in \mathcal{T}), \quad \text{on } (\Omega, \mathcal{F}, P),$$

where $\mathcal{T}$ is discrete or continuous, e.g.

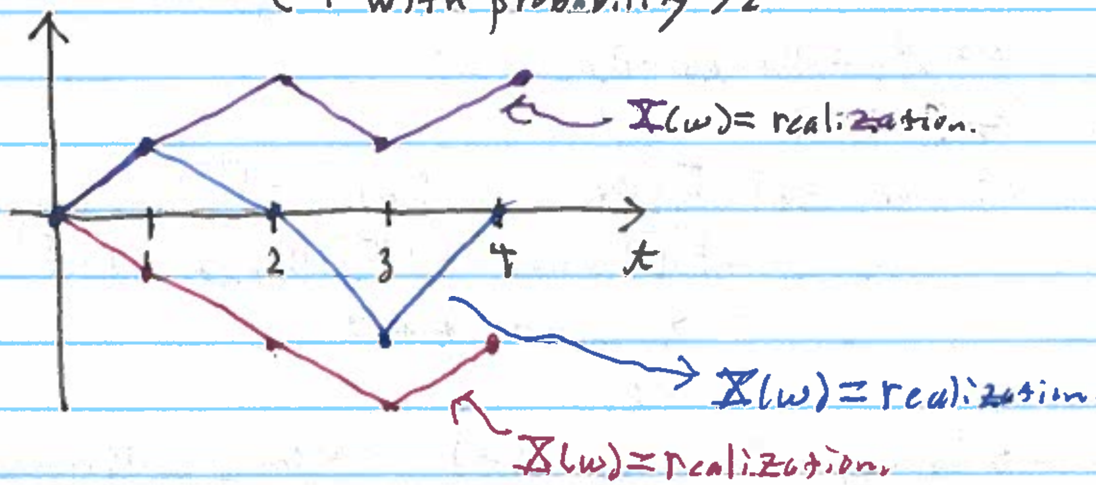
$$\mathcal{T} = \{0, 1, 2, 3, \dots\} \text{ or } \mathcal{T} = [0, \infty).$$

Examples

1. $\mathcal{T} = \{0, 1, 2, 3, \dots\}$

$\Rightarrow X_0 = 0$

$$X_{i+1} - X_i = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$



2. $\mathcal{T} = [0, \infty)$

$X_0 = 0$

$$X_{t+\Delta t} - X_t \sim \underbrace{N(0, \sigma^2 \Delta t)}$$

Gaussian with mean 0 and variance $\sigma^2 \Delta t$.

Definition - A Gaussian process $X = (X_t, t \in T)$ is a stochastic process whose finite-dimension distributions are Gaussian, i.e., for all $n \in \mathbb{N}$ and any $t_1 < \dots < t_n$, $(X_1, \dots, X_n)$ is Gaussian.
The distribution is determined by

$$m(t) = \mathbb{E}[X_t]$$
$$C(s,t) = Cov[X_t, X_s].$$

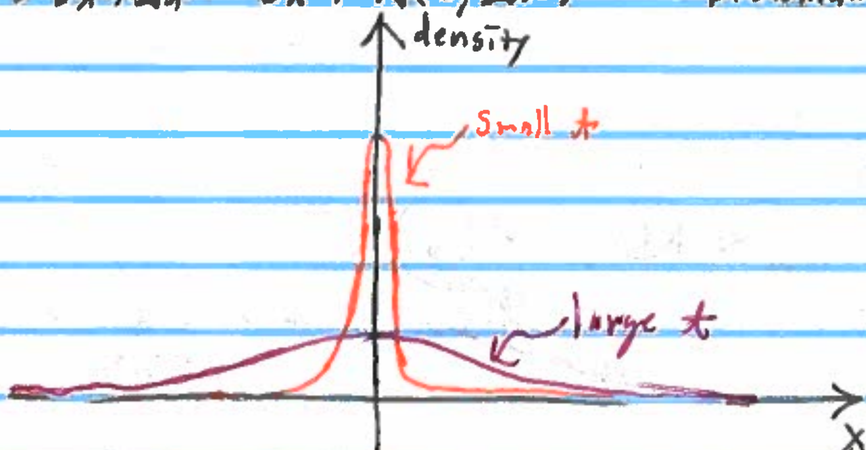
Examples:

1. Brownian Motion:

$(B_t, t \geq 0)$ is the Gaussian process with mean $m(t) = \mathbb{E}[B(t)] = 0$ and $Cov(B_t, B_s) = \mathbb{E}[B_t B_s] = \min(s,t)$.

$$\Rightarrow Z_t = B_{t+\Delta t} - B_t$$
$$\Rightarrow \mathbb{E}[Z_t] = 0$$
$$\begin{aligned}\Rightarrow Var[Z_t] &= \mathbb{E}[(B(t+\Delta t) - B_t)^2] - \mathbb{E}[B(t+\Delta t) - B_t]^2 \\ &= \mathbb{E}[B_{t+\Delta t}^2] - 2\,\mathbb{E}[B_{t+\Delta t} B_t] + \mathbb{E}[B_t^2] \\ &= t + \Delta t - 2t + t \\ &= \Delta t.\end{aligned}$$

$$\Rightarrow B_{t+\Delta t} = B_t + N(0, \Delta t) \longrightarrow \text{Brownian Motion}$$



2. Brownian Motion with Drift

Let $X_t = \sigma B_t + \mu t$.

$\Rightarrow \mathbb{E}[X_t] = \mu t$

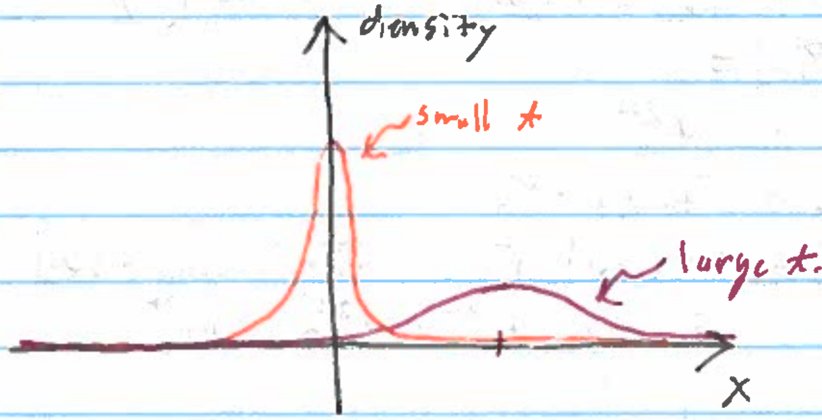
$$Cov[X_t, X_s] = Cov[\sigma B_t + \mu t, \sigma B_s + \mu s]$$
$$= \sigma^2 Cov(B_t, B_s) + \mu\sigma Cov(t, B_s)$$
$$+ \mu s\, Cov(B_t, s) + \mu^2 Cov(s, t)$$

$* Cov(t, B_s) = \mathbb{E}[tB_s] - \mathbb{E}[t]\mathbb{E}[B_s] = 0$

$\Rightarrow Cov[X_t, X_s] = \sigma^2 \min(s,t).$

Therefore,

$$Var(X_t) = \sigma^2 t.$$



3. Brownian Bridge

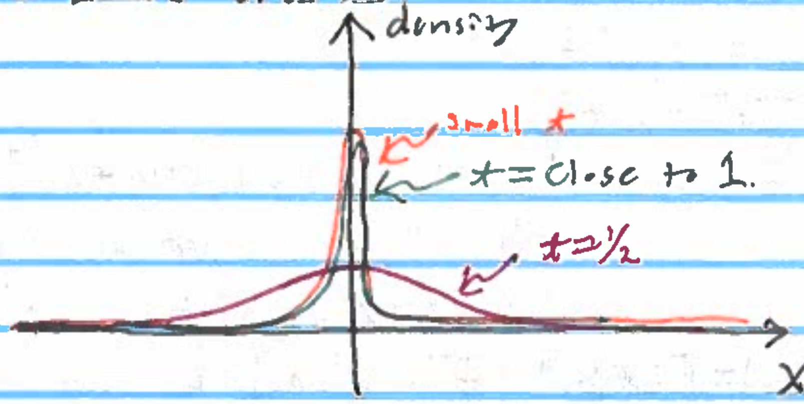
$$Z_t = B_t - tB_1, \quad t \in [0,1]$$

$\Rightarrow \mathbb{E}[Z_t] = 0$

$\Rightarrow Cov[Z_t, Z_s] = Cov[B_t - tB_1, B_s - sB_1]$

$$= Cov[B_t, B_s] + st\, Cov[B_1, B_1]$$
$$- t\, Cov[B_1, B_s] - s\, Cov[B_t, B_1]$$
$$= \min(s,t) + st - t - s$$
$$= \begin{cases} s(1-t) & \text{if } s \le t \\ t(1-s) & \text{if } t \le s \end{cases}$$

$\Rightarrow Var[Z_t] = t(1-t)$



4. Fractional Brownian Motion

Let $0 < H < 1$

Define: $Cov(Y_s, Y_t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$

$\mathbb{E}[Y_t] = 0$

$\Rightarrow Var[Y_t] = t^{2H}$

$\Rightarrow$ Let $Z_t = Y_{t+\Delta t} - Y_t$

$$\Rightarrow Var[Z_t] = Cov[Y_{t+\Delta t} - Y_t, Y_{t+\Delta t} - Y_t]$$
$$= (t+\Delta t)^{2H} + t^{2H} - |\Delta t|^{2H}$$