Lecture 8: Properties of Brownian Motion

Theorem: A process B_t defined on $(\Omega, \mathcal{F}, P)$ has the distribution of a standard Brownian motion on $[0,\infty)$ if and only if the following hold:

(i) $B_0 = 0$

(ii) For any $s<t$, the increment $B_t - B_s$ is Gaussian with mean 0 and variance $t-s$.

(iii) For any $n \in \mathbb{N}$, $0 \le t_1 < t_2 < \ldots < t_n < \infty$ the increments $(B_{t_2} - B_{t_1})(B_{t_3} - B_{t_2}) \cdots (B_{t_n} - B_{t_{n-1}})$ are independent.

(iv) The path $t \to B_t(w)$ is continuous with probability 1.

Example:

What is the probability that $B_1 > 0$ and $B_2 > 0$.

$$\mathrm{Cov}(B_1, B_1) = 1$$
$$\mathrm{Cov}(B_1, B_2) = 1$$
$$\mathrm{Cov}(B_2, B_2) = 2$$

The covariance matrix is therefore

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow C^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

The joint density is

$$f(b_1, b_2) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left(\frac{-\underline{b}^T C^{-1} \underline{b}}{2}\right)$$

$$= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}(2b_1^2 - 2b_1 b_2 + b_2^2)\right)$$

Example:

1. Reflection at time s: The process $(-B_t, t \geq 0)$ is a Brownian motion. More generally, for any $s \geq 0$ the process

$$\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq s \\ B_s - (B_t - B_s) & \text{if } t > s \end{cases}$$

is a Brownian motion.

2. Scaling: For any $a > 0$, the process $(\frac{1}{\sqrt{a}} B_{at}, t \geq 0)$ is a Brownian motion.

3. Time reversal: The process $(B_1 - B_{1-t}, t \in [0,1])$ is a Brownian motion on $[0,1]$.

proof:

1. • $\tilde{B}_0 = 0$

• For $t < t'$ we have that

$$Cov(\tilde{B}_t, \tilde{B}_{t'}) = \begin{cases} Cov(B_t, B_{t'}) & \text{if } t' \leq s \\ Cov(B_t, B_s - (B_{t'} - B_s)) & \text{if } t \leq s \leq t' \\ Cov(B_s - (B_t - B_s), B_s - (B_{t'} - B_s)) & \text{if } s \leq t. \end{cases}$$

$$= \begin{cases} t & \text{if } t' \leq s \\ Cov(B_t, B_s) - Cov(B_t, B_{t'}) + Cov(B_t, B_s) & \text{if } t \leq s \leq t' \\ Cov(B_s, B_s) - Cov(B_s, B_t - B_s) - Cov(B_s, B_{t'} - B_s) & \text{if } s \leq t \\ + Cov(B_t - B_s, B_{t'} - B_s) & \end{cases}$$

$$= \begin{cases} t & \text{if } t' \leq s \\ t & \text{if } t \leq s \leq t' \\ s - 0 - 0 + Cov(B_t, B_{t'}) - Cov(B_s, B_{t'}) & \text{if } s \leq t \\ - Cov(B_t, B_s) + Cov(B_t, B_s) & \end{cases}$$

$$= t.$$

• $\mathbb{E}[\tilde{B}_t] = 0$, $Var[\tilde{B}_t] = t$.

2. • $\frac{1}{\sqrt{a}} B_{a\cdot 0} = 0$

• $Cov\left(\frac{1}{\sqrt{a}} B_{at}, \frac{1}{\sqrt{a}} B_{as}\right) = \frac{1}{a} Cov(B_{at}, B_{as})$

$$= \frac{1}{a} \min(at, as)$$

$$= \min(t, s).$$

• Clearly $\frac{1}{\sqrt{a}} B_{at}$ is Gaussian since it is a linear transformation of a Gaussian

3. Let $Z_t = B_1 - B_{1-t}$.

• $Z_0 = 0$

• $Cov(Z_t, Z_s) = Cov(B_1 - B_{1-t}, B_1 - B_{1-s})$

$$= Cov(B_1, B_1) - Cov(B_{1-t}, B_1) - Cov(B_{1-s}, B_1) + Cov(B_{1-t}, B_{1-s})$$

$$= 1 - (1-t) - (1-s) + \min(1-t, 1-s)$$

$$= \begin{cases} t & \text{if } t \leq s \\ s & \text{if } s \leq t \end{cases}$$

$$= \min(t, s).$$

• Clearly Z_t is Gaussian since it is a linear transformation of Gaussian random variables.