Lecture 9: Conditional Expectation

Example:

Let X, Y be two discrete random variables with

$$P(X=1)=\tfrac{1}{2}, \quad P(Y=1)=\tfrac{1}{2}$$
$$P(X=0)=0 \quad P(Y=0)=0$$

Let $Z = X + Y$.

- What is $\mathbb{E}[Z \mid X=0]$?

$$\Rightarrow \mathbb{E}[Z \mid X=0] = \sum_{j=0}^{2} j\, P(Z=j \mid X=0)$$

$$= 1P(Z=1 \mid X=0) + 2P(Z=2 \mid X=0)$$

$$= 1\frac{P(Z=1, X=0)}{P(X=0)} + 2\frac{P(Z=2, X=0)}{P(X=0)}$$

$$= 1 \cdot \frac{1/4}{1/2} + 2 \cdot \frac{0}{1/2}$$

$$= 1/2.$$

- What is $\mathbb{E}[Z \mid X=1]$?

$$\Rightarrow \quad \mathbb{E}[Z \mid X=1] = 1 \cdot P(Z=1 \mid X=1) + 2P(Z=2 \mid X=1)$$

$$= \frac{P(Z=1, X=1)}{P(X=0)} + 2\frac{P(Z=2, X=1)}{P(X=0)}$$

$$= \frac{1/4}{1/2} + 2 \cdot \frac{1/4}{1/2}$$

$$= 1/2 + 1$$

$$= 3/2.$$

$$\Rightarrow \mathbb{E}[Z \mid X](\omega) \doteq \begin{cases} 1/2 & \text{if } X(\omega)=0 \\ 3/2 & \text{if } X(\omega)=1 \end{cases}$$

Observation:

$$\mathbb{E}[g(\mathbb{X})Z] = \mathbb{E}[g(\mathbb{X})\,\mathbb{E}[Z_1|\mathbb{X}]]$$

→ In terms of $\mathbb{X}$, $\mathbb{E}[Z|\mathbb{X}]$ is the same as $\mathbb{X}$.

proof:

$$\mathbb{E}[g(\mathbb{X})Z] = \sum_{i=0}^{1}\sum_{j=0}^{2} j\,g(i)\,P(\mathbb{X}=i, Z_1=j)$$

$$= \sum_{i=0}^{1} g(i) \sum_{j=0}^{2} j\,P(\mathbb{X}=i, Z_1=j)$$

$$= \sum_{i=0}^{1} g(i)P(\mathbb{X}=i) \sum_{j=0}^{2} j\,P(\mathbb{X}=i, Z_1=j)/P(\mathbb{X}=i)$$

$$= \sum_{i=0}^{1} g(i)P(\mathbb{X}=i)\,\mathbb{E}[Z_1|\mathbb{X}]$$

$$= \mathbb{E}[g(\mathbb{X})\,\mathbb{E}[Z_1|\mathbb{X}]].$$

Definition - Let $\mathbb{X}, \mathbb{Y}$ be continuous random variables with joint PDF $f(x,y)$. Then

$$\mathbb{E}[\mathbb{Y}|\mathbb{X}] = h(\mathbb{X}),$$

where

$$h(x) = \int_{-\infty}^{\infty} y\,f(x,y)\,dy \Big/ \int_{-\infty}^{\infty} f(x,y)\,dy.$$

Definition - Let X, Y be integrable random variables on $(\Omega, \mathcal{F}, P)$. The conditional expectation of Y given X is the random variable denoted by $\mathbb{E}[Y|X]$ with the following two properties:

(a) There exists a function $h: \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[Y|X] = h(X)$

(b). For any random variable of the form $g(X)$,

$$\underbrace{\mathbb{E}[g(X)Y]}_{\text{average over joint density}} = \mathbb{E}[\underbrace{g(X)\mathbb{E}[Y|X]}_{\text{average over values of } X \text{ only}}]$$

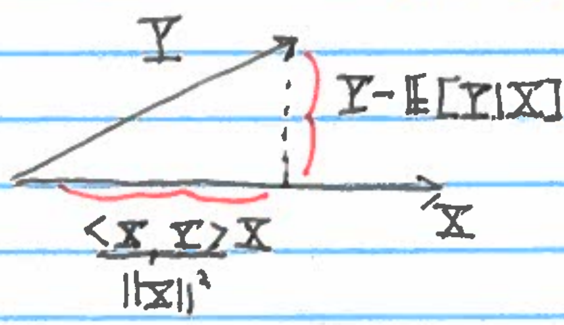
Corollary - The random variable $Y - \mathbb{E}[Y|X]$ is orthogonal to $g(X)$.

proof:

$$\mathbb{E}[g(X)Y] = \mathbb{E}[g(X)\mathbb{E}[Y|X]]$$
$$\Rightarrow \mathbb{E}[g(X)Y] - \mathbb{E}[g(X)\mathbb{E}[Y|X]] = 0$$
$$\Rightarrow \mathbb{E}[g(X)(Y - \mathbb{E}[Y|X])] = 0$$
$$\Rightarrow \langle g(X), Y - \mathbb{E}[Y|X] \rangle = 0.$$

only true if $Y \in L^2$.

Theorem - $\mathbb{E}[Y|X] = \dfrac{\mathbb{E}[XY]}{\mathbb{E}[X^2]} X = \dfrac{\langle X, Y \rangle}{\|X\|^2} X$



proof

$$\left\langle X, Y - \frac{\langle X, Y \rangle}{\|X\|^2} X \right\rangle = \langle X, Y \rangle - \frac{\langle X, Y \rangle \langle X, X \rangle}{\|X\|^2} = 0.$$

Which proves that $X, Y - \langle X, Y\rangle / \|X\|^2 X$ are independent and thus $g(X)$ and $Y - \langle X, Y\rangle / \|X\|^2 X$ are independent as well.

Example:

Consider the Gaussian vector $(B_{1/2}, B_1)$.

- What is $\mathbb{E}[B_1 \mid B_{1/2}]$?

$$\mathbb{E}[B_1 \mid B_{1/2}] = \frac{\mathbb{E}[B_1 B_{1/2}]}{\mathbb{E}[B_{1/2}^2]} B_{1/2} = \frac{1/2}{1/2} B_{1/2} = B_{1/2}.$$

- What is $\mathbb{E}[B_{1/2} \mid B_1]$?

$$\mathbb{E}[B_{1/2} \mid B_1] = \frac{\mathbb{E}[B_{1/2} B_1]}{\mathbb{E}[B_1^2]} B_1 = \frac{1/2}{1} B_1 = 1/2\, B_1.$$