Chapter 4: Flows on the Circle

To fully describe a differential equation

$$\dot{x} = f(x)$$

one must also define the space the solution curves live on.

Examples

1. $\mathbb{R}$ - position of a car on a straight track
2. $\mathbb{R}^+$ - population growth
3. S^1 (unit circle) - motion on circular track, angles

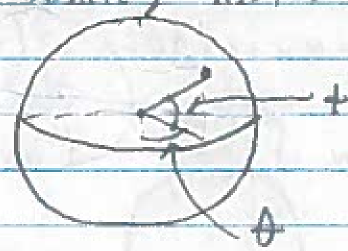
θ → This labels a point

$\theta = 0, 2\pi$ are two labels for the same point.

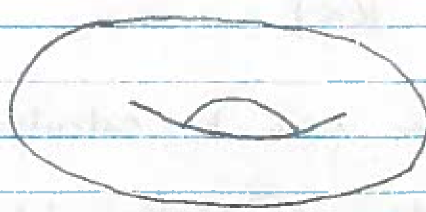
$\dot{\theta} = \theta^2$ cannot be vector field on S^1.

A vector field on S^1 must satisfy $\dot{\theta} = f(\theta + 2n\pi)$ for all $n \in \mathbb{Z}$.

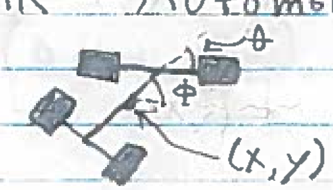
4. S^2 (unit sphere) - motion on earth



5. $S^1 \times S^1$ (torus) - Two angles



6. $S^1 \times S^1 \times \mathbb{R}^2$ - Automobile



L7

We will concentrate on the circle.

Example:

$$\dot{\theta} = \omega - a\sin(\theta)$$

→ Phase locking, θ is a phase difference.

Rescale:

$$\tau = \omega t$$

$$\Rightarrow \frac{d\theta}{d\tau} = 1 - \alpha\sin(\theta), \quad \alpha = a/\omega$$

Fixed points:

$$\theta = \sin^{-1}\left(\frac{1}{\alpha}\right) \quad \text{and} \quad \theta = \pi - \sin^{-1}\left(\frac{1}{\alpha}\right)$$

if $\alpha \geq 1$. Stability analysis

$$\frac{df}{d\theta} = -\alpha\cos(\theta)$$

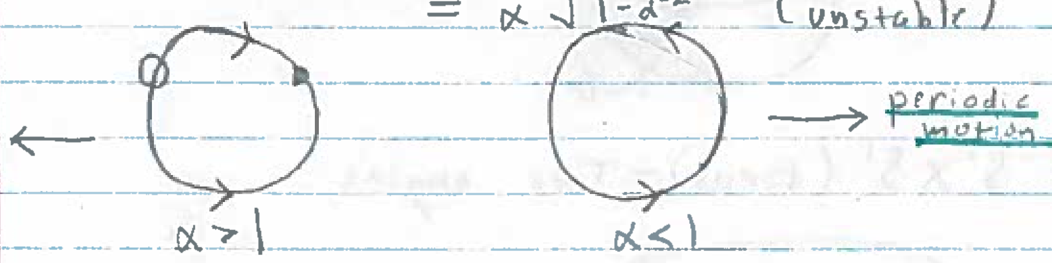
$$\Rightarrow \left.\frac{df}{d\theta}\right|_{\sin^{-1}(\frac{1}{\alpha})} = -\alpha\cos\left(\sin^{-1}\left(\tfrac{1}{\alpha}\right)\right)$$

$$= -\alpha\sqrt{1-\alpha^{-2}} \quad \text{(stable)}$$

$$\left.\frac{df}{d\theta}\right|_{\pi-\sin^{-1}(\frac{1}{\alpha})} = -\alpha\cos\left(\pi - \sin^{-1}\left(\tfrac{1}{\alpha}\right)\right)$$

$$= \alpha\sqrt{1-\alpha^{-2}} \quad \text{(unstable)}$$

locked phase

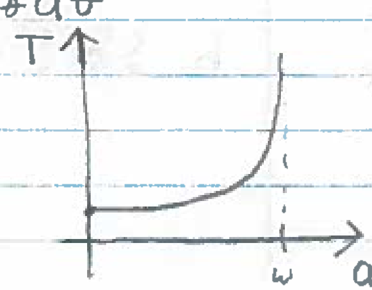


periodic motion

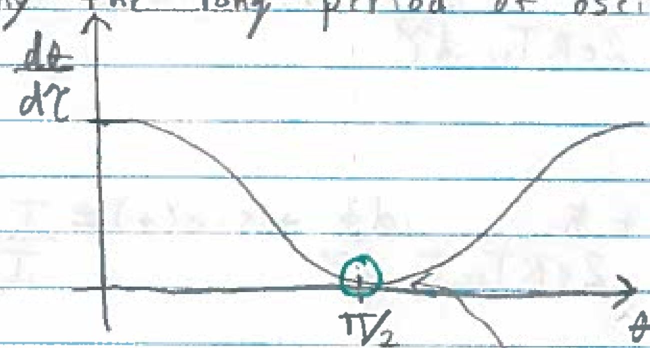
The period of oscillation can be calculated:

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\theta}\,d\theta = \int_0^{2\pi} \frac{1}{\omega - a\sin\theta}\,d\theta$$

$$\Rightarrow T = \frac{2\pi}{\sqrt{\omega^2 - a^2}} \quad (\text{If } \omega > a)$$



Why the long period of oscillation?



$a > 1$, and $a \approx 1$.

$\frac{d\theta}{d\tau} \approx 0 \Rightarrow$ very slow dynamics.

Near $\theta = \pi/2$, $1 - \alpha \sin\theta \approx 1 - \alpha + \alpha \frac{(\theta - \pi/2)^2}{2}$

$$\Rightarrow \frac{d\theta}{d\tau} \approx 1 - \alpha + \alpha \frac{\theta^2}{2}$$

The passage through the slow period can be estimated:

$$T_{passage} \approx \int_{-\infty}^{\infty} \frac{1}{1 - \alpha + \frac{\alpha}{2} x^2} dx = \frac{\sqrt{2}\,\pi}{\sqrt{\alpha}\sqrt{1-\alpha}} \approx \frac{\sqrt{2}\,\pi}{\sqrt{1-\alpha}}$$

This is known as a square root scaling law.

L8

Example:

$$\frac{\hbar C}{2e} \ddot{\phi} + \frac{\hbar}{2eR} \dot{\phi} + I_c \sin\phi = I \quad \text{(superconducting Josephenson junction)}$$

$[I] = I$ (current)

$\left[\frac{\hbar}{2eR}\right] = \frac{I}{T}$

$\left[\frac{\hbar C}{2e}\right] = \frac{I}{T^2}$

Rescale by $\tau = \frac{t}{T_{sc}}$

$$\Rightarrow \frac{\hbar C}{2eT_{sc}^2}\frac{d^2\phi}{d\tau^2} + \frac{\hbar}{2eRT_{sc}}\frac{d\phi}{d\tau} + I_c \sin(\phi) = I$$

Normalizing

$$\frac{\hbar C}{2eT_{sc}^2 I_c}\frac{d^2\phi}{d\tau^2} + \frac{\hbar}{2eRT_{sc}I_c}\frac{d\phi}{d\tau} + \sin(\phi) = \frac{I}{I_c}$$

We want

$$\frac{\hbar}{2eRT_{sc}I_c} = O(1)$$

$$\Rightarrow \boxed{T_{sc} = \frac{\hbar}{2eRI_c}}$$

We obtain the system:

$$\frac{\cancel{\hbar} C}{\cancel{2e}\cdot\frac{\hbar^{\cancel{2}}}{2^2\cancel{e^2}R^2I_c^2}I_c}\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin(\phi) = \gamma$$

$$\Rightarrow \beta\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin(\phi) = \gamma$$

$$\boxed{\beta = \frac{2eI_cR^2C}{\hbar}}$$

If $\beta << 1$ we have the system

$$\frac{d\phi}{d\tau} + \sin(\phi) = \gamma$$

$$\Rightarrow \boxed{\frac{d\phi}{d\tau} = \gamma - \sin(\phi)}$$

$\gamma < 1$ $\gamma > 1$ ← periodic motion

What is the average velocity?

$$\left\langle \frac{d\phi}{dt} \right\rangle = \frac{1}{T_{per}}\int_0^{T_{per}} \frac{d\phi}{dt}\,dt = \frac{1}{T_{per}}\,2\pi$$

where

$$T_{per} = \int_0^{T_{per}} dt$$

$$= \int_0^{2\pi} \frac{1}{\gamma - \sin(\phi)} dt$$

$$= \frac{2\pi}{\sqrt{\gamma^2 - 1}}$$

$$\Rightarrow \left\langle \frac{d\phi}{dt} \right\rangle = \sqrt{\gamma^2 - 1}, \text{ if } \gamma > 1.$$

Example:

$\dot{\phi} = \Omega$ entrainment frequency

$\dot{\theta} = \omega + A\sin(\phi - \theta)$

natural frequency   frequency update.

Let $\varphi = \phi - \theta$

$\Rightarrow \dot{\varphi} = \Omega - \omega - A\sin(\varphi)$

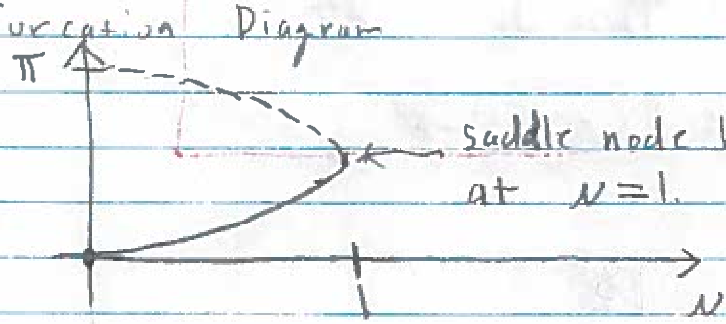
Rescale

$\tau = At$

and let $\mu = \frac{\Omega - \omega}{A}$. This gives the system

$$\frac{d\varphi}{d\tau} = \mu - \sin(\varphi)$$

Bifurcation Diagram



saddle node bifurcation at $\mu = 1$.

When $0<\nu\leq 1$ we get entrainment. The range of entrainment is

$$\omega - A \leq \Omega \leq \omega + A$$

This interval is the range of entrainment.

If $\nu>1$, the period of phase drift is given by:

$$T_{drift} = \int_0^{T_{drift}} dt = \int_0^{2\pi} \frac{dt}{d\phi} d\phi = \int_0^{2\pi} \frac{d\phi}{\Omega-\omega-A\sin(\phi)}$$

$$\Rightarrow \boxed{T_{drift} = \frac{2\pi}{\sqrt{(\Omega-\omega)^2-A^2}}}$$

The time of passage through a bottleneck is given by Taylor expanding near $\varphi=\pi/2$:

$$\dot{\varphi} \approx \nu - \sin(\varphi) \approx \nu - 1 + \tfrac{1}{2}(\varphi - \pi/2)^2$$

$$\Rightarrow \frac{d\varphi}{d\tau} \approx \nu - 1 + \frac{\varphi^2}{2}$$

$$\Rightarrow \boxed{\tau_{passage} \approx \int_{-\infty}^{\infty} \frac{1}{\nu - 1 + \frac{x^2}{2}} dx = \frac{\sqrt{2}}{\sqrt{\nu-1}} \pi.}$$

The average velocity is

$$\boxed{\left\langle \frac{d\varphi}{d\tau} \right\rangle = \frac{1}{T_{drift}} \int_{T_1}^{T_1+T_{drift}} \frac{d\phi}{d\tau} d\tau = \sqrt{(\Omega-\omega)^2-A^2}}$$