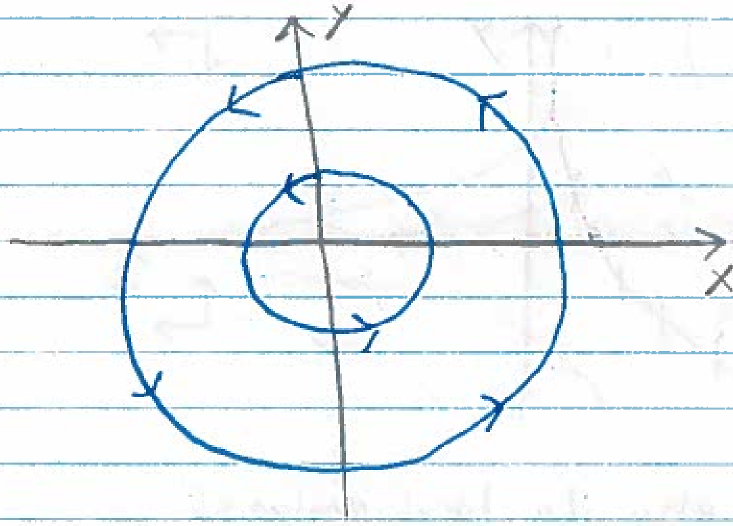


$Re(\lambda)=0$
(center).

Notation.

1. A is called hyperbolic if $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
2. We say that the fixed point $x=0$ of $\dot{x}=Ax$ is
   a.) an attractor if $\vec{x}(t)\to 0$ as $t\to\infty \Rightarrow Re(\lambda_1,\lambda_2)<0$
   b.) a repellor if $\vec{x}(t)\to 0$ as $t\to-\infty \Rightarrow Re(\lambda_1,\lambda_2)>0$
   c.) a saddle if $\lambda_1<0<\lambda_2$
   d.) non hyperbolic if $Re(\lambda_1)=0$ or $Re(\lambda_2)=0$

Example:

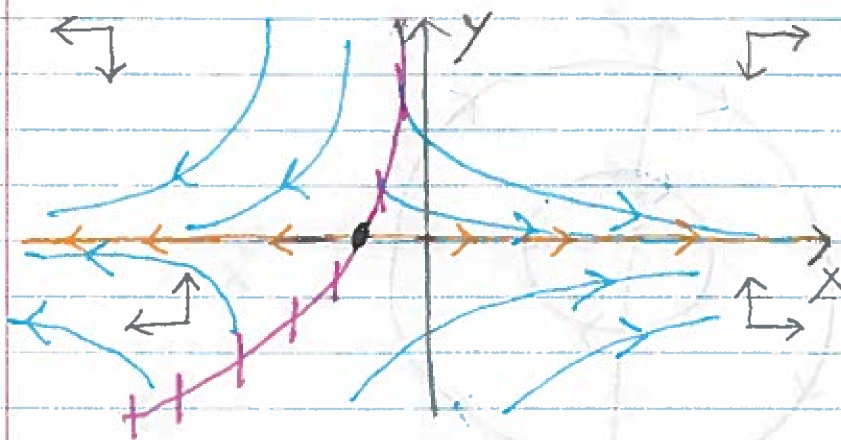
Sketch the phase portrait of
$$\begin{cases} \dot{x}=x+e^{-y} = f(x,y) \\ \dot{y}=-y \quad = g(x,y). \end{cases}$$

Fixed points: $y=0$ and $x=-1$.

Nullclines:

a.) $-x=e^{-y} \Rightarrow y=-\ln(-x)$ N1: $\frac{dx}{dt}=0$

b.) $y=0$ N2: $\frac{dy}{dt}=0$



We can also do local analysis:

linearization about $(-1,0)$:

$$f(x,y) = f(\vec{x}^*) + \nabla f\big|_{\vec{x}^*} \cdot (\vec{x}-\vec{x}^*) + \tfrac{1}{2}(\vec{x}-\vec{x}^*)^T H(\nabla^2 f)(\vec{x}-\vec{x}^*)$$
$$g(x,y) = g(\vec{x}^*) + \nabla g\big|_{\vec{x}^*} (\vec{x}-\vec{x}^*) + \tfrac{1}{2}(\vec{x}-\vec{x}^*)^T H(\nabla^2 f)(\vec{x}-\vec{x}^*).$$

Near equilibrium

$$F(\vec{x}-\vec{x}^*) \approx J(F)(\vec{x}-\vec{x}^*)$$
$$J(E) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Changing variables $\tilde{x} = x - x^*$, $\tilde{y} = y - y^*$

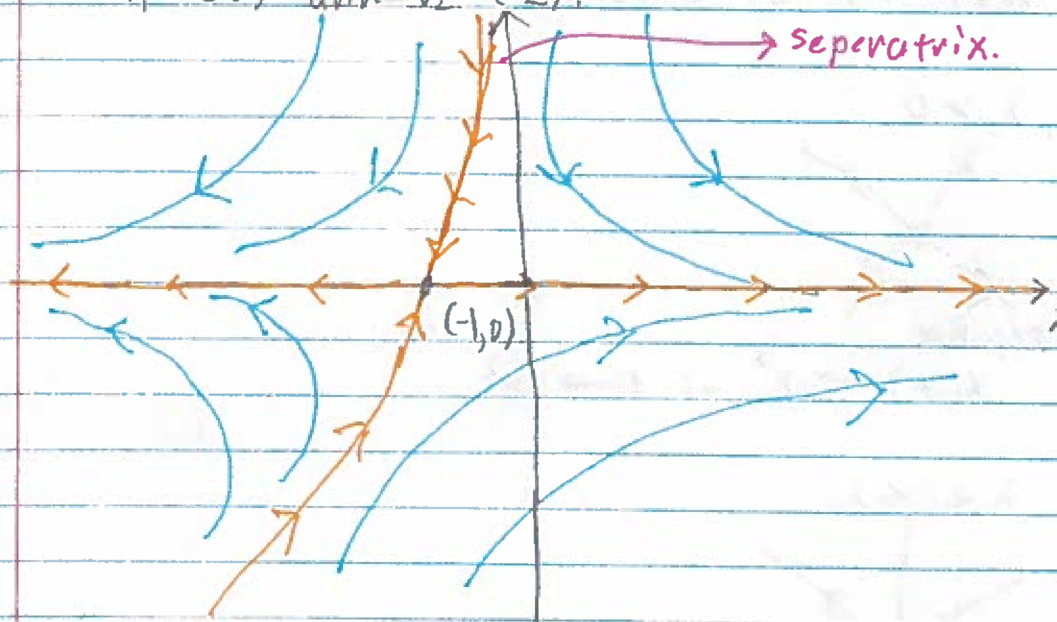
$$F(\tilde{\vec{x}}) \approx J(F)\big|_{\vec{x}^*} \tilde{\vec{x}}$$

Locally, the solution is
$$\tilde{\vec{x}} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$
where λ_1, λ_2 are the eigenvalues of $J\big|_{\vec{x}^*}$.

For our problem:

$$J(F)\big|_{(-1,0)} = \begin{pmatrix} 1 & -e^{-y} \\ 0 & -1 \end{pmatrix}\Bigg|_{(-1,0)} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$ with eigenvectors $\vec{v}_1 = \binom{1}{0}$ and $\vec{v}_2 = \binom{1}{2}$.



Phase Plane Summary:

1. Draw Nullclines
2. Draw direction arrows $\uparrow\!\!\rightarrow$
3. Find equilibrium
4. Calculate Jacobian
5. Determine eigenvalues and eigenvectors.

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\Rightarrow$ hyperbolic points

*Determine fast direction.

$\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ $\Rightarrow$ spirals

* Determine stable or unstable

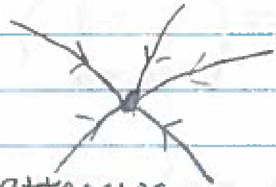
λ_1, λ_2 pure Imaginary

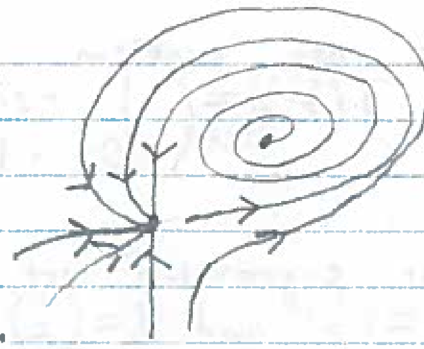
$\Rightarrow$ Need more work.

λ_1 or $\lambda_2 = 0$

$\Rightarrow$ Need more work

a.) $\lambda_1, \lambda_2 < 0$





attractor

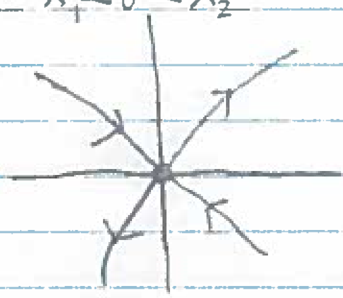
$x(t) \to x^* \text{ as } t \to \infty.$

b.) $\lambda_1, \lambda_2 > 0$

repeller

$x(t) \to x^* \text{ as } t \to -\infty.$

c.) $\lambda_1 < 0 < \lambda_2$



Saddle

Example

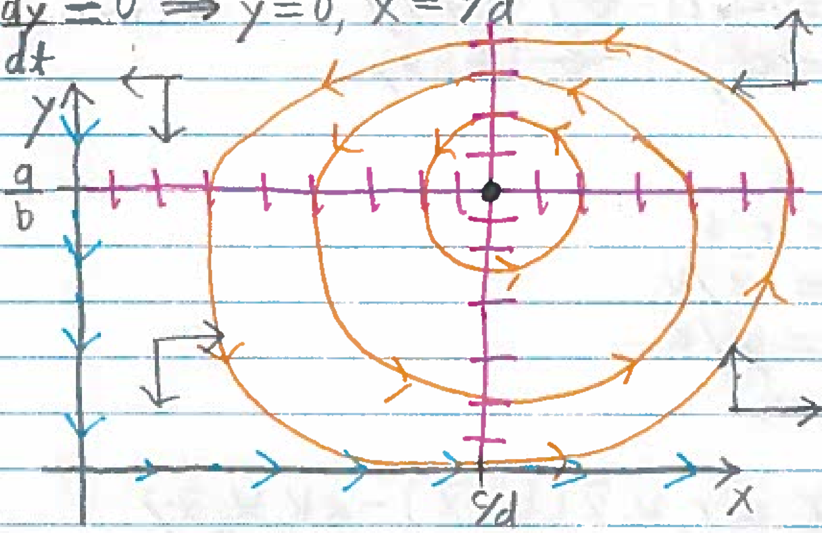
Predator-Prey

$$\dot{x} = ax - bxy$$
$$\dot{y} = -cy + dxy$$

Null Clines:

$$\frac{dx}{dt} = 0 \Rightarrow x = 0,\ y = a/b$$

$$\frac{dy}{dt} = 0 \Rightarrow y = 0,\ x = c/d$$



Eigenvalues analysis:

$$J = \begin{pmatrix} a-by & -bx \\ dy & -c+dx \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \Rightarrow \begin{array}{ll} \lambda_1 = a, & \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = -c, & \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{array}$$

$$J(c/d, a/b) = \begin{pmatrix} 0 & -bc/d \\ da/b & 0 \end{pmatrix}$$

The eigenvalues are pure imaginary $\Rightarrow$ cannot deduce anything.
However,

$$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{(dx-c)}{(a-by)}$$

$$\Rightarrow \int\left(\frac{a}{y} - b\right) dy = \int\left(-\frac{c}{x} + d\right)$$

$$\Rightarrow a\ln(y) - by = -c\ln(x) + dx + C$$

$$a\ln(y) - by + c\ln(x) - dy = C$$

$$y^a e^{-by} x^c e^{-dy} = C$$

This describes a family of closed curves $\Rightarrow$ the solution curves

Example:

Logistic growth omnivores and prey.

$$\dot{x} = r_1 x\left(1-\frac{x}{K_1}\right) - \alpha x y \qquad \text{(Invasive Predators)}$$
$$\dot{y} = r_2 y\left(1-\frac{y}{K_2}\right) + \beta x y$$

Rescale

$$\tau = r_1 t$$
$$\bar{x} = x/K_1$$
$$\bar{y} = y/K_2$$

Therefore,

$$r_1 K_1 \frac{d\bar{x}}{d\tau} = r_1 K_1 \bar{x}(1-\bar{x}) - \alpha K_1 K_2 \bar{x}\bar{y}$$
$$r_1 K_2 \frac{d\bar{y}}{d\tau} = r_2 K_2 \bar{y}(1-\bar{y}) + \beta K_1 K_2 \bar{x}\bar{y}$$

$$\Rightarrow \frac{d\bar{x}}{d\tau} = \bar{x}(1-\bar{x}) - \gamma \bar{x}\bar{y}$$
$$\frac{d\bar{y}}{d\tau} = \eta\, \bar{y}(1-\bar{y}) + \delta \bar{x}\bar{y}$$

Equilibrium points:

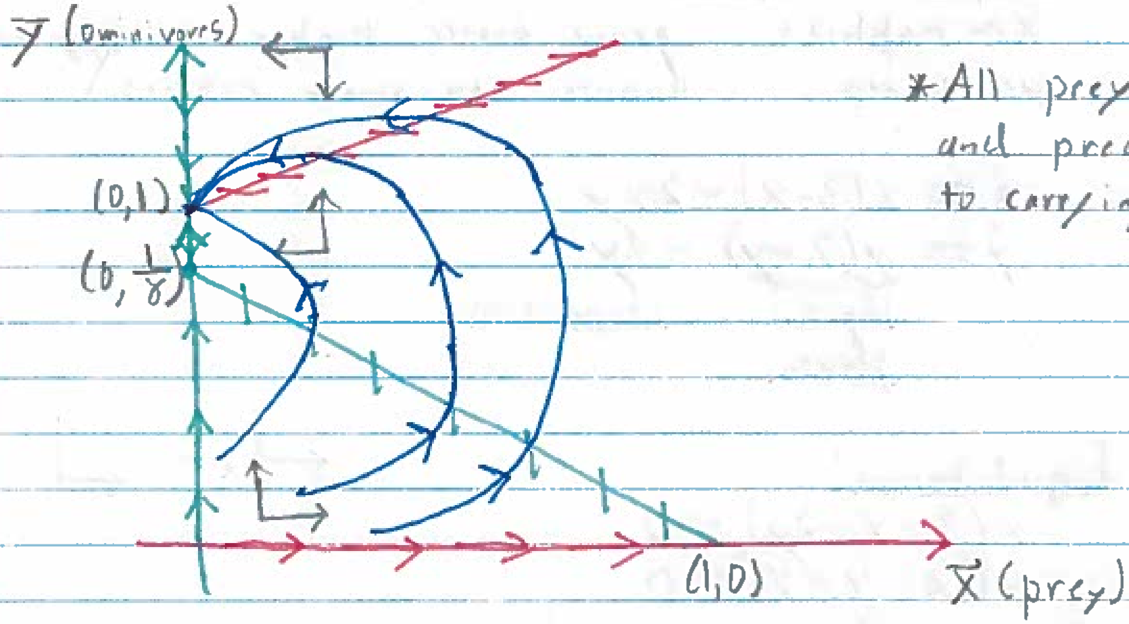
$$0 = \bar{x}(1-\bar{x}) - \gamma \bar{x}\bar{y}$$
$$0 = \eta \bar{y}(1-\bar{y}) + \delta \bar{x}\bar{y}$$

$\Rightarrow (\bar{x},\bar{y}) = (0,0)$ is one equilibrium. There are others.

Null clines:

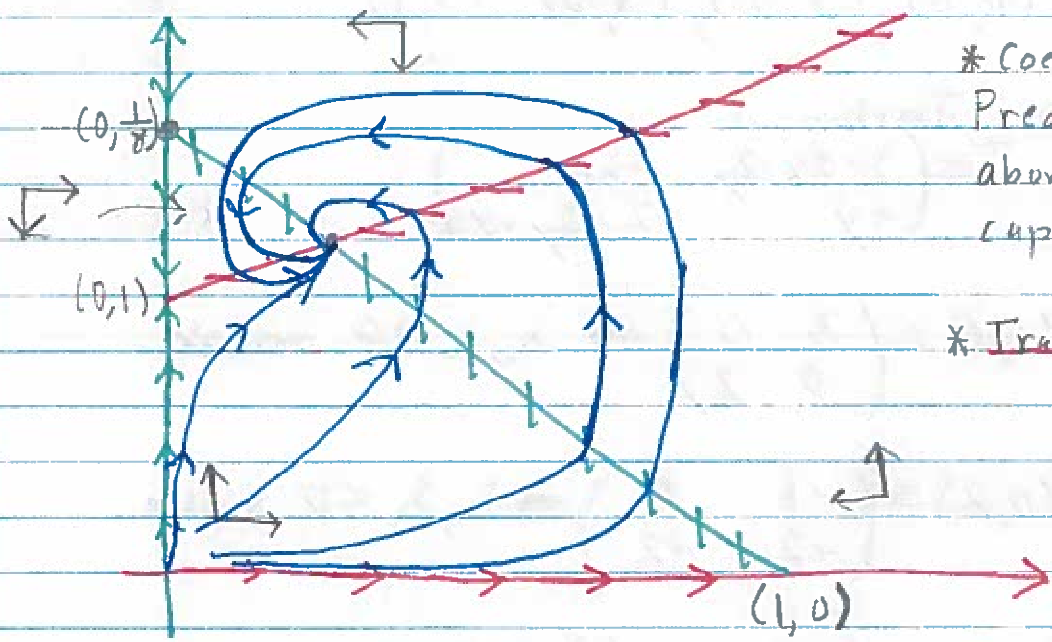
N1: $\bar{y} = \frac{1-\bar{x}}{\gamma}$, $\bar{x}=0$: $\frac{dx}{d\tau}=0$

N2: $\bar{x} = \frac{\eta}{\delta}(\bar{y}-1)$, $\bar{y}=0$: $\frac{dy}{d\tau}=0$



* All prey die off and predators go to carrying capacity.

Omnivores take over



* Coexistence. Predators live above carrying capacity.

* Transcritical bifurcati

Point of intersection:

$$\bar{x}^* = \frac{(1-\gamma)\eta}{\gamma\delta+\eta} \qquad \bar{y}^* = \frac{\delta+\eta}{\gamma\delta+\eta}$$

The Jacobian is a mess but it is complex valued at the critical point

$\Longrightarrow$ Stable spiral

Example: (Competition)

$x \sim$ rabbits , grow faster higher carrying capacity

$y \sim$ sheep , hooves can smash rabbits.

$$\dot{x} = x(3-x) - 2xy$$
$$\dot{y} = \underbrace{y(2-y)}_{\text{logistic growth}} - \underbrace{xy}_{\text{competition}}$$

Equilibrium:

$$x(3-x-2y) = 0$$
$$y(2-y-x) = 0$$

We get four equilibrium:

$(0,0)$, $(0,2)$, $(3,0)$, $(1,1)$

The Jacobian is

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{pmatrix}$$

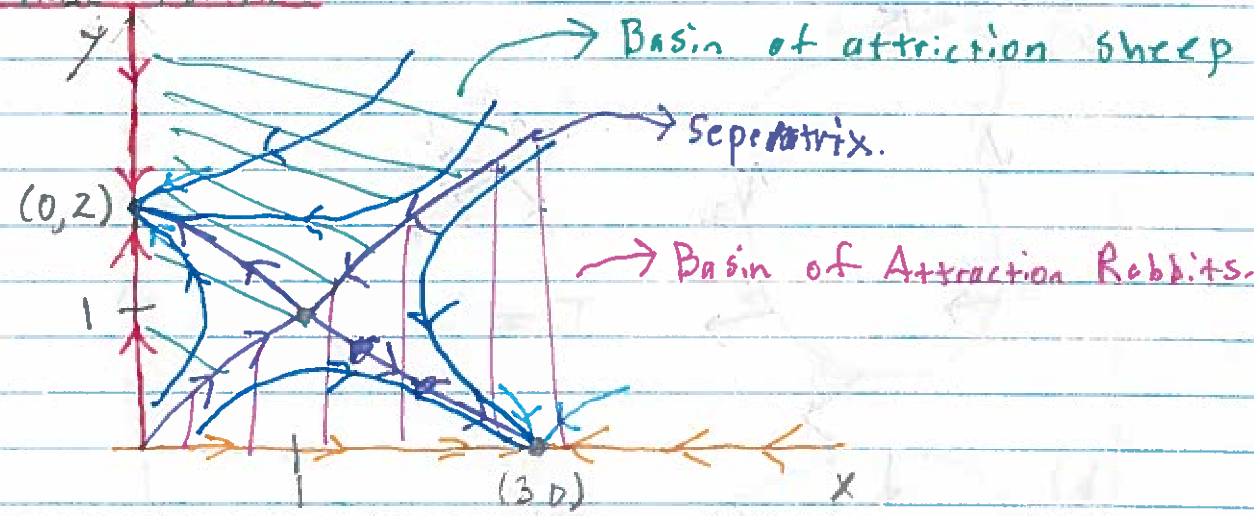
$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 > 0 \quad \text{unstable}$$

$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 < 0 \quad \text{stable}$$

$$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 < 0 \quad \text{stable}$$

$$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 = -1 \pm \sqrt{2} \quad \text{saddle}$$

Phase Portrait



Index Theory

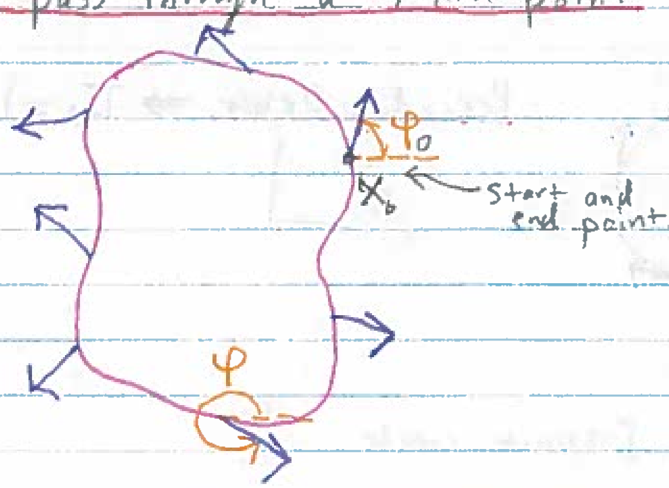
How can we be sure no periodic orbits exist?

Consider

$$\dot{\vec{x}} = F(\vec{x})$$

with $F: \mathbb{R}^2 \to \mathbb{R}^2$ continuously differentiable.

Take a closed curve Γ with no self intersections, that does not pass through a fixed point

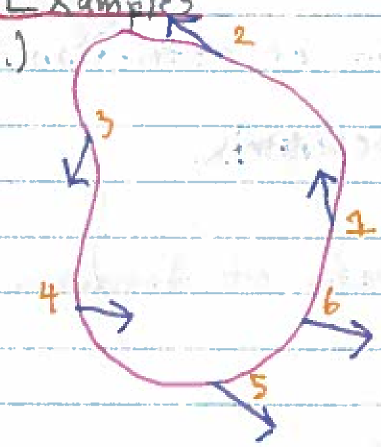


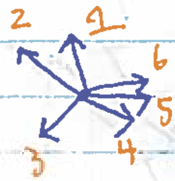
1. Start at U_0 traverse Γ counterclockwise and take angle φ of $F(\vec{x}) \to$ this angle changes continuously as Γ is traversed
2. After one pass we again end up at U_0 with an angle $\varphi_1 = \varphi_0 \pm 2\pi n$; $n \in \mathbb{Z}$

$$\boxed{I_\Gamma = \frac{1}{2\pi}(\varphi_1 - \varphi_0)}$$

Examples

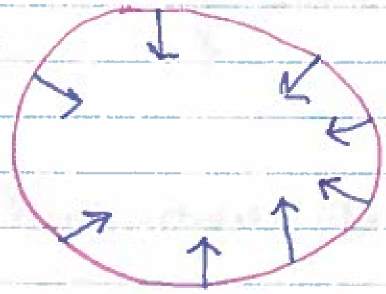
a.)





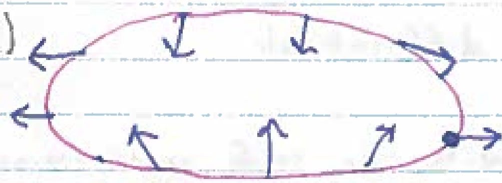
$I_p = 1$

b.)



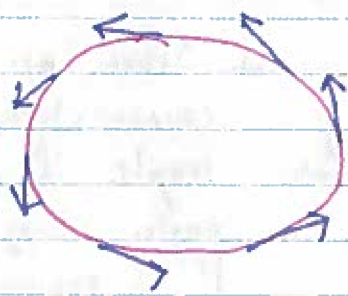
$I_p = 1$

c.)



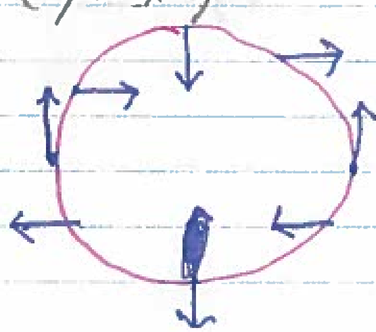
$I_p = -1$

d.)



Periodic Orbit $\Rightarrow I_p = 1$

e.) $\begin{cases} \dot{x} = x^2 y \\ \dot{y} = x^2 - y^2 \end{cases}$, Γ = unit circle



$I_p = 0$

Properties of the Index

1. If Γ can be deformed continuously into $\tilde{\Gamma}$ without passing through any equilibrium points then

$$I_\Gamma = I_{\tilde{\Gamma}}$$

~~proof~~:
I_Γ varies continuously as Γ is deformed, but I_Γ is integer valued.

2. If Γ does not contain any fixed points then $I_\Gamma = 0$

~~proof~~:
Property 1 implies we can shrink Γ to a point without changing the index.

3. If we replace $F(\vec{x})$ by $F(-\vec{x})$ the index is not changed.

~~proof~~:
Each angle is replaced by $\varphi + \pi$, hence $\varphi_1 - \varphi_0$ is the same.

4. The index of a periodic orbit is one.
5. If $F(\vec{x})$ is deformed continuously without creating any fixed points on Γ, I_Γ stays the same.

Theorem - Assume F is continuously differentiable. Inside each periodic orbit, there is at least one equilibrium.

~~proof~~
Follows from items 2 and 4.

Index of isolated fixed point - Let $\vec{x}^*$ be an isolated fixed point of $\dot{\vec{x}} = F(\vec{x})$. Define

$I(\vec{x}^*)$ = index of simple closed curve that encloses x^* and no other fixed points

$I(\vec{x}^*)$ is well defined by property 1.

Consequences

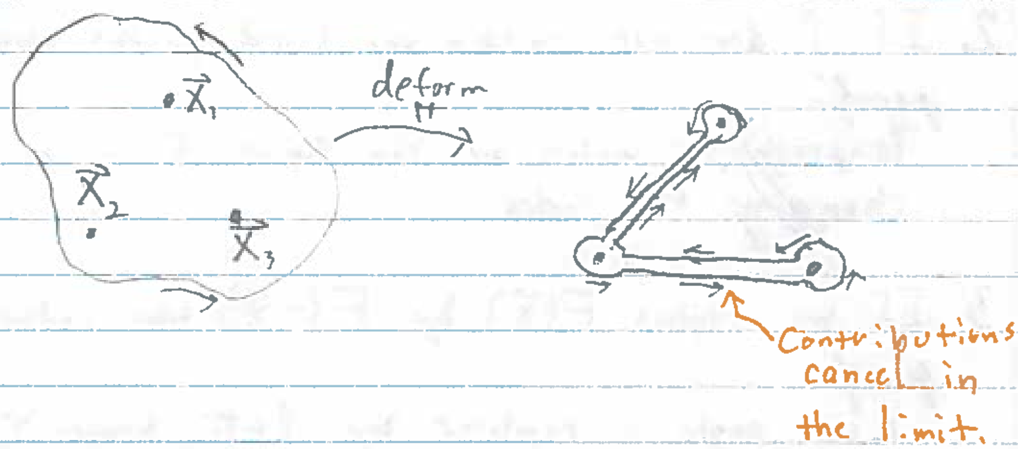
1. If $\vec{x}^*$ is an attractor or repellor then $I(\vec{x}^*)=1$
2. If $\vec{x}^*$ is a saddle point then $I(\vec{x}^*)=-1$.

~~proof~~:

Follows from examples b and c and properties 1, 3, 5.

Theorem - If Γ is a closed simple curve that contains n isolated fixed points $\vec{x}_1, \dots, \vec{x}_n$ then $I_\Gamma = I(\vec{x}_1) + \dots + I(\vec{x}_n)$

~~proof~~:



Corollary: A periodic orbit must enclose fixed points whose indices sum to +1.

Omnivore example:

The index of all the fixed points is -2.

Sheep and Rabbits:

The index of all the fixed points is 0.

Example:

$\dot{x} = y$

$\dot{y} = -x + (1-x^2-y^2)y$

Null-clines:

N1: $\dot{x}=0$, $y=0$

N2: $\dot{y}=0$, $y(1-x^2-y^2)=x$

$y - yx^2 - y^3 = x$

$y - y^3 = x + x^2 y$

$\Rightarrow \frac{y-y^3}{1+y} = x + x^2$ → Not very informative.

Idea:

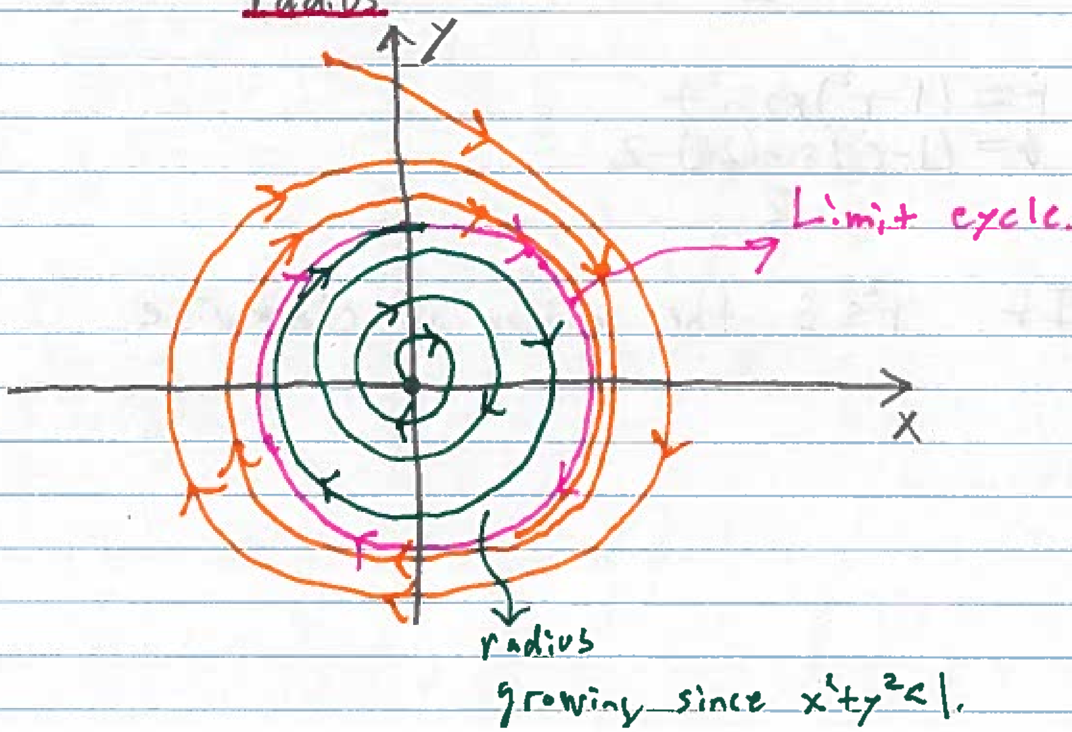
$\ddot{x} + x = (1-x^2-y^2)y$

$\Rightarrow \dot{x}\ddot{x} + x\dot{x} = (1-x^2-y^2)\dot{x}\cdot y$

$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{x}^2 + x^2) = (1-x^2-y^2)\dot{x}^2$

$\Rightarrow \frac{1}{2}\frac{d}{dt}(y^2+x^2) = (1-x^2-y^2)y^2$

radius



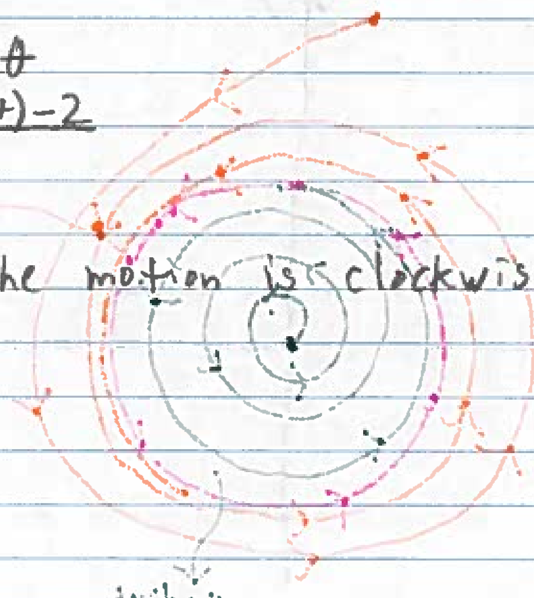
Convert to polar coordinates:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$
$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} \qquad y = r\sin\theta$$
$$\Rightarrow \dot{r} = (x\dot{x} + y\dot{y})/r$$
$$= \frac{\cancel{xy - xy} + (1-r^2)\cdot y^2}{r}$$
$$= (1-r^2)\cdot r\sin^2\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\dot{\theta} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{x\dot{y} - y\dot{x}}{x^2}\right)$$
$$= \frac{x\dot{y} - y\dot{x}}{r^2}$$
$$= \frac{-x^2 + (1-r^2)xy - y^2}{r^2}$$
$$= -1 + (1-r^2)\sin\theta\cos\theta$$
$$= -1 + \frac{(1-r^2)\sin(2\theta)}{2}$$
$$= \frac{(1-r^2)\sin 2\theta - 2}{2}$$

$$\dot{r} = (1-r^2)\, r\sin^2\theta$$
$$\dot{\theta} = \frac{(1-r^2)\sin(2\theta) - 2}{2}.$$

If $r^2 \leq 3$ the motion is clockwise.



Conservative Systems:

Inertial Systems of the form:

$$\ddot{x} = F(x)$$

A first integral can be found as follows:

$$\dot{x}\ddot{x} = \dot{x}F(x)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{x}^2) = \frac{dx}{dt}\left(\frac{-dV}{dx}\right), \quad \text{(For any solution curve}$$

where $V(x) = -\int_{x_0}^{x} F(x)\,dx$ (x_0 can be chosen arbitrarily)

$$\Rightarrow \frac{d}{dt}\left(\frac{\dot{x}(t)^2}{2} + V\right) = 0$$

For any solution curve there is a constant E such that

$$\boxed{\frac{\dot{x}(t)^2}{2} + V(x(t)) = E}$$

We can also write as a system:

$$\dot{x} = v \qquad \frac{v^2}{2} + V(x) = E$$
$$\dot{v} = F(x)$$

phase portrait $\Longleftrightarrow$ contour plot

Theorem - A conservative system cannot have any attractor or repellors.

~~proof~~:

Suppose there exists $(x^*\ v^*)$ that is an attracting point with a basin of attraction A. Then, for all $(x_1, v_1), (x_2, v_2) \in$ it follows that $E(x_1, v_1) = E(x_2, v_2)$. Since

$$E(x_1, v_1) = \lim_{t\to\infty} E(x_1(t), v_1(t))$$

$$= E(x^*, v^*(t))$$

$$= \lim_{t\to\infty} E(x_2(t), v_2(t))$$

$$= E(x_2, v_2).$$

Therefore, E must be constant in entire basin of attraction which we preclude by definition. ∎

Example

$\ddot{x} + \sin(x) = 0$

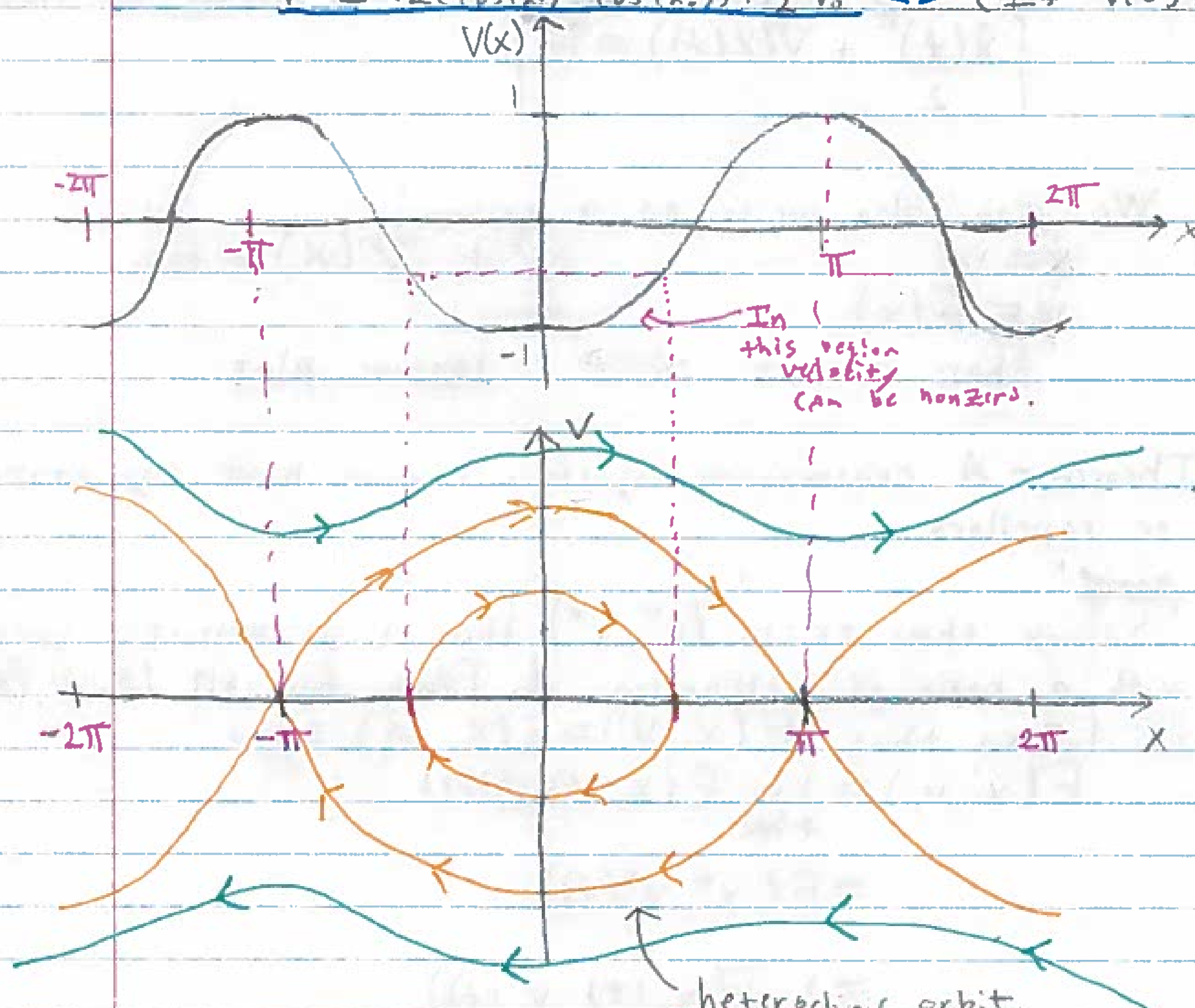
$V(x) = -\cos(x)$

$E = \frac{1}{2}v^2 - \cos(x)$

$\Rightarrow v = \pm\sqrt{2E + 2\cos(x)}$

$v = \pm\sqrt{2(\cos(x) - \cos(x_0))} \Leftrightarrow$ (If $v(0) = 0$)

$v = \pm\sqrt{2(\cos(x) - \cos(x_0)) + \frac{1}{2}v_0^2} \Leftrightarrow$ (If $v(0) \neq 0$).



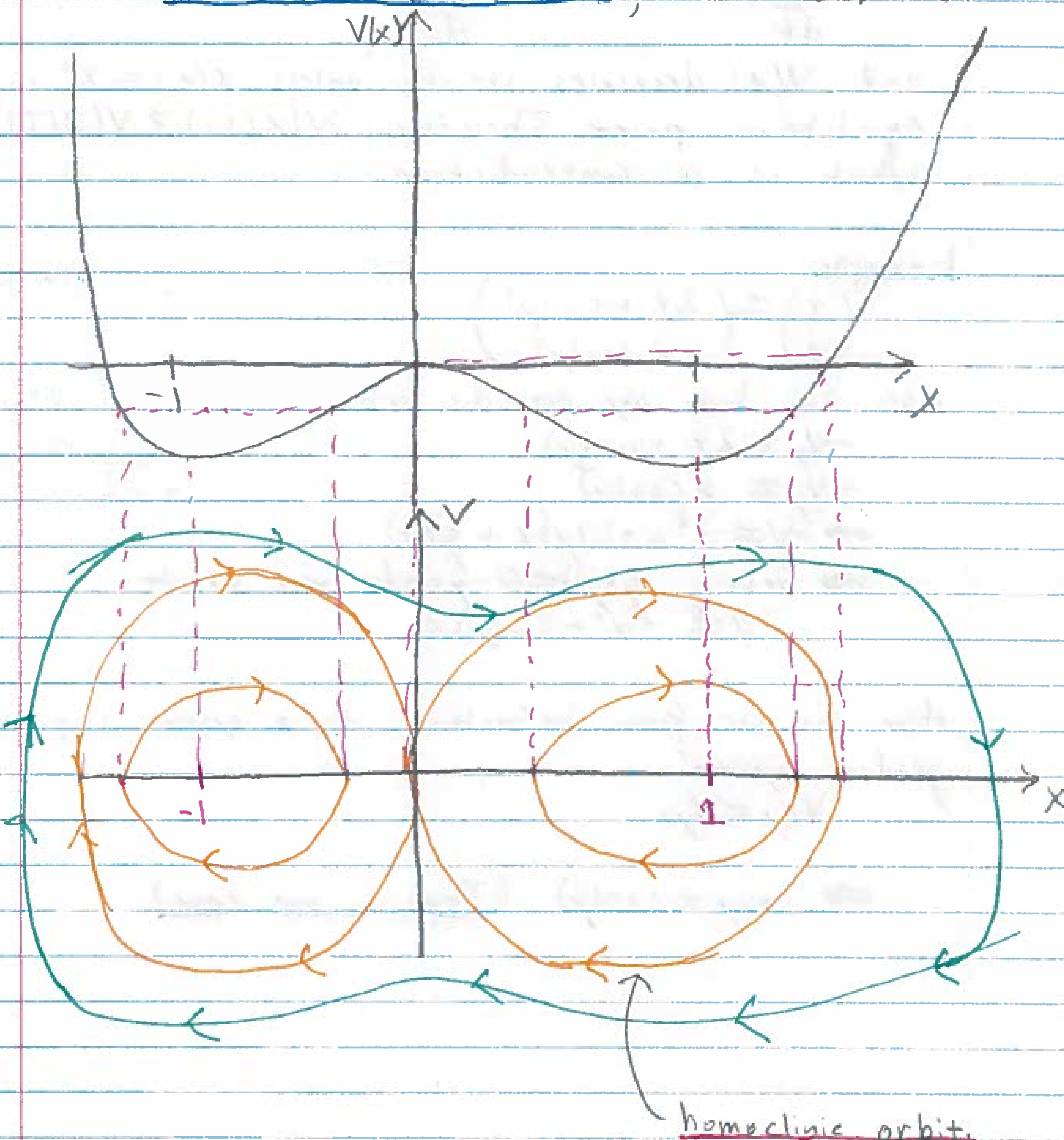
Example

$\ddot{x} = x - x^3$

$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$

$E = \frac{1}{2}v^2 - \frac{x^2}{2} + \frac{x^4}{4}$

$\Rightarrow v = \pm\sqrt{x^2 - \frac{x^4}{2} - x_0^2 + \frac{x_0^4}{2}}$, (If $v(0) = 0$).



G Gradient Systems.

$$\dot{\vec{x}} = -\nabla V, \quad \text{where} \quad V: \mathbb{R}^2 \to \mathbb{R}$$

Lemma - Gradient systems cannot have closed orbits.

proof:

Let $x(t)$ be a closed orbit with period T. Then,

$$\frac{d}{dt} V(x(t)) = \nabla V \cdot \frac{d\vec{x}}{dt} = -|\nabla V|^2 \leq 0.$$

and $V(t)$ decreases strictly unless $x(t) = x^*$ is an equilibrium point. Therefore, $V(x(0)) > V(x(T))$ which is a contradiction.

Example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2x + \sin(y) \\ x\cos(y) \end{pmatrix}$$

does not have any periodic orbits.

$$-V_x = 2x + \sin(y)$$
$$-V_y = x\cos(y)$$
$$\Rightarrow -V = x^2 + \sin(y)x + g(y)$$

$\Rightarrow$ Setting $g(y) = 0$ yields the result

$$V = -x^2 - \sin(y)x$$

How can we know beforehand if a system is potentially a gradient system?

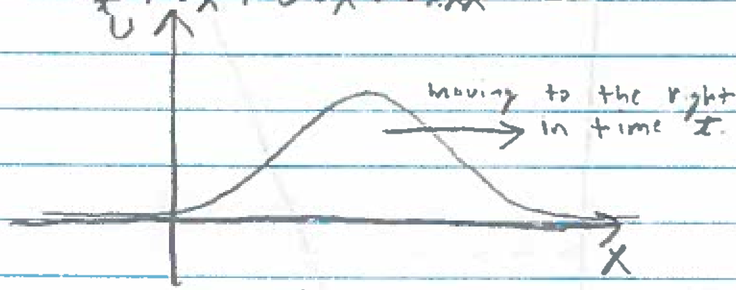
$$V_{xy} = V_{yx}$$

$$\Rightarrow \cos y = \cos(y) \quad \text{(True in our case)}$$

Solitons

Shallow water waves in a narrow canal

$$u_t + u_x + u u_x + u_{xxx} = 0$$



Let $z = x - ct$

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{d}{dz} = \frac{d}{dz}$$

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{d}{dz} = -c\frac{d}{dz}$$

$$\Rightarrow (1-c)\frac{du}{dz} + \frac{1}{2}\frac{d}{dz}(u^2) + u_{zzz} = 0$$

$$\Rightarrow (1-c)u + \frac{1}{2}u^2 + u_{zz} = \nu$$

$$\Rightarrow u_{zz} = -(1-c)u - \frac{1}{2}u^2 + \nu$$

This is a conservative system with potential:

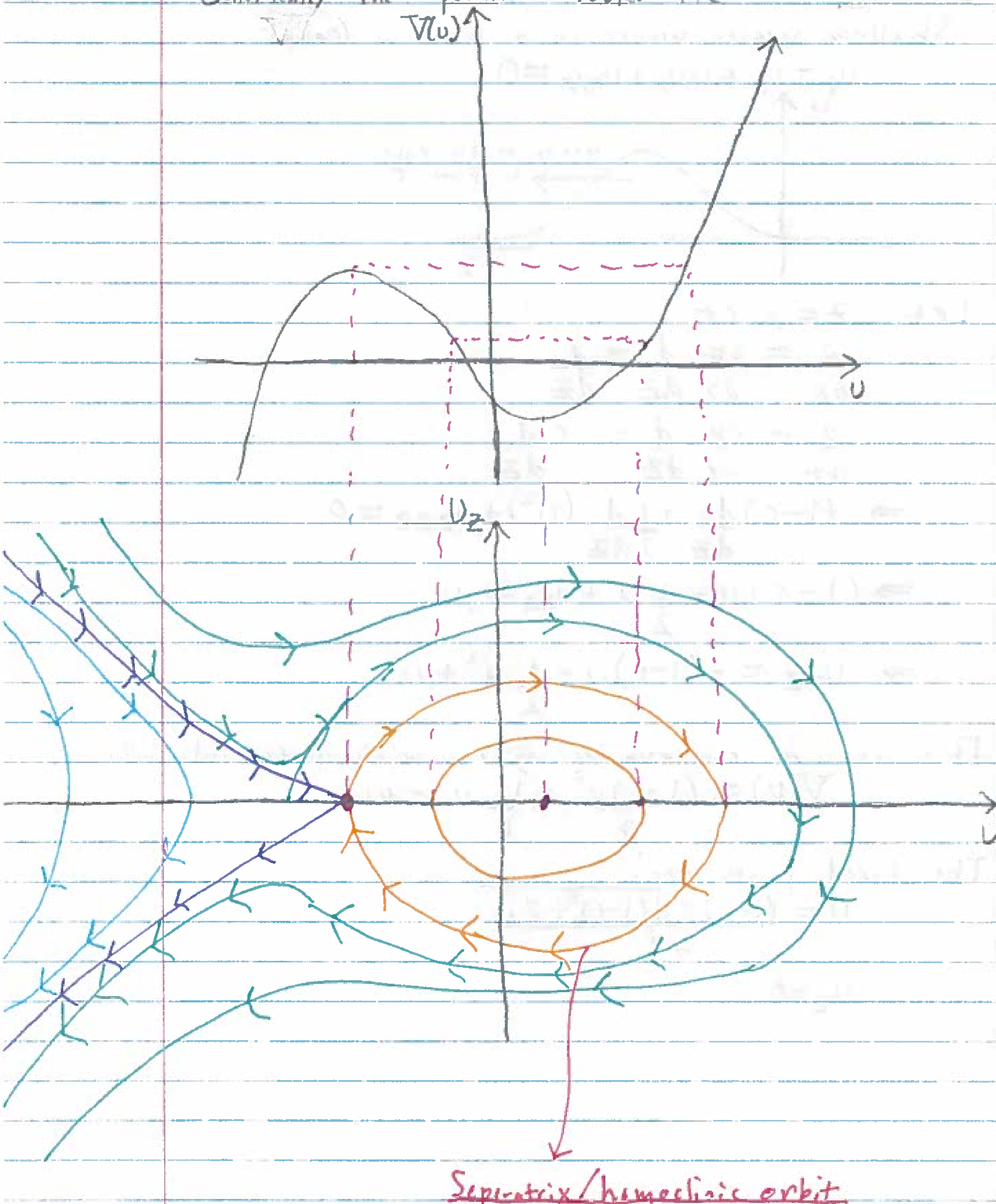
$$V(u) = \frac{(1-c)u^2}{2} + \frac{1}{6}u^3 - \nu u.$$

The fixed points are:

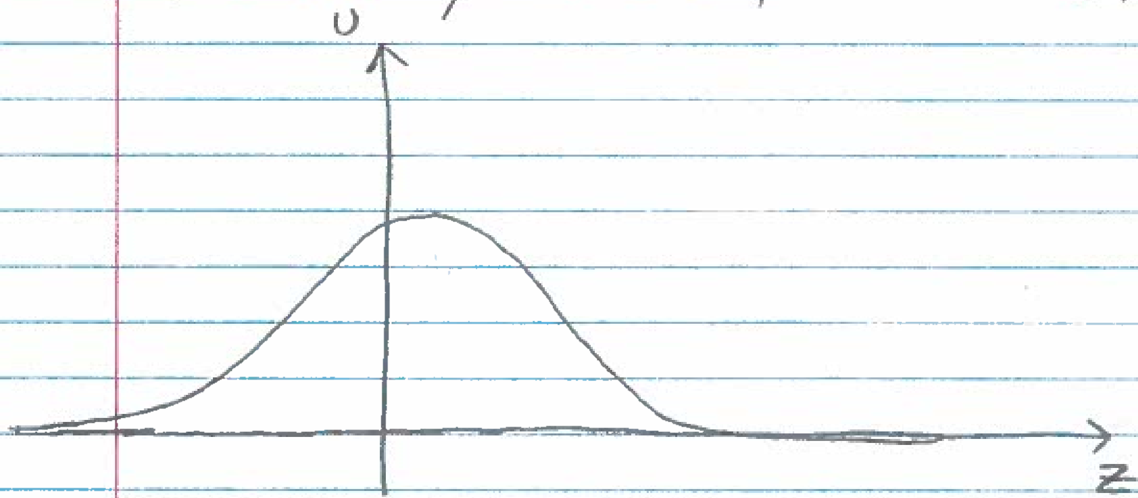
$$u = \frac{(1-c) \pm \sqrt{(1-c)^2 + 2\nu}}{-1}$$

$$u_z = 0$$

Generically the potential looks like:



The travelling wave corresponds to the seperatrix / homoclinic



Sketch of the solution.

Lyapunov Functions

$\dot{\vec{x}} = F(\vec{x})$

A continuously differentiable function $L : \mathbb{R}^2 \to \mathbb{R}$ is called a Lyapunov function if $L(x(t))$ strictly decreases along each solution of $\dot{\vec{x}} = F(\vec{x})$ that is not an equilibrium.

Lemma - If $\dot{\vec{x}} = F(\vec{x})$ admits a Lyapunov function, then it cannot have any periodic orbits.

Example

$\ddot{x} + \alpha \dot{x} = g(x)$, $\alpha > 0$.

Let $V(x) \triangleq -\int_{x_0}^{x} g(x)\,dx$. Then,

$$\frac{1}{2}\frac{d(\dot{x}^2)}{dt} + \alpha \dot{x}^2 = -\frac{d}{dt}(V(x(t)))$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V(x(t))\right) = -\alpha \dot{x}^2 < 0.$$

The function $L(v, x) = \frac{1}{2}v^2 + V$ is a Lyapunov function.

Summary:

1. $\ddot{x} = -\frac{dV}{dx} \to$ conservative $E(x, \dot{x})$ is conserved $\}$ Many periodic orbits.
2. $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla V \to$ gradient system, V decreases along solutions
3. $\ddot{x} + \alpha \dot{x} = -\frac{dV}{dx} \to E(x, \dot{x})$ decreases

No periodic solutions