Homework #3.

#3.5.7

Consider the logistic equation:

$$\dot{N} = rN(1 - N/K),$$
$$N(0) = N_0.$$

Nondimensionalize this equation in two natural ways.

Solution:

a.) Let $\tau = rt$, $x = N/K$. Then,

$$\frac{d}{dt}N = \frac{dN}{d\tau}\frac{d\tau}{dt} = r\frac{d}{d\tau}(Kx) = rK\frac{dx}{d\tau}.$$

Consequently,

$$rK\frac{dx}{d\tau} = rKx(1-x)$$
$$\Rightarrow \frac{dx}{d\tau} = x(1-x)$$
$$x(0) = \frac{N(0)}{K} = \frac{N_0}{K} = x_0$$

b.) Let $\tau = rt$, $x = N/N_0$. Then

$$rN_0\frac{dx}{d\tau} = rN_0x\left(1 - \frac{N_0}{K}x\right)$$
$$\Rightarrow \frac{dx}{d\tau} = x\left(1 - \frac{x}{\alpha}\right),$$
$$x(0) = 1,$$

where $\alpha = N_0/K$.

#3.7.5

Consider the model

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

where the k's are positive constants. Analyze the model when $s_0 = 0$.

Solution:

Let $x = g/k_4$. Then,

$$k_4\dot{x} = k_1 s_0 - k_2 k_4 x + \frac{k_3 x^2}{1 + x^2}.$$

Let $\tau = k_3 t$. Consequently, $\frac{d}{dt} = k_3 \frac{d}{d\tau}$ and therefore

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2},$$

where $s = k_1 s_0 / k_3$ and $r = k_2/k_3$. Hence, if $s_0 = 0$ then

$$\frac{dx}{d\tau} = -rx + \frac{x^2}{1+x^2}. \qquad (*)$$

The fixed points for this system satisfy:

$$x = 0, \quad \frac{x}{1+x^2} - r = 0.$$

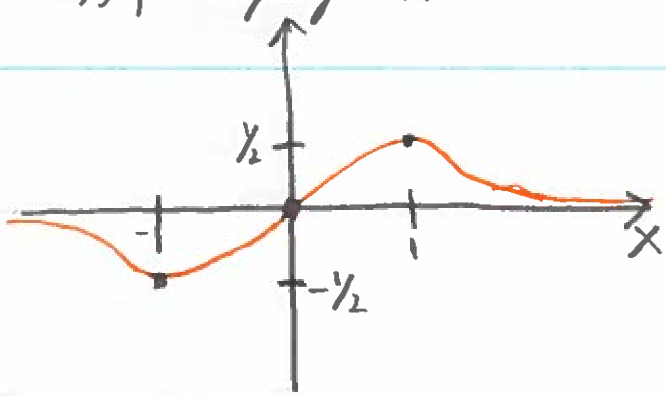
Let $f(x) = x/(1+x^2)$. Therefore,

$$f'(x) = \frac{1+x^2-2x^2}{(1+x^2)} = \frac{1-x^2}{(1+x^2)}.$$

which implies that $f(x)$ has critical points at $x = \pm 1$. Letting, $g(x) = \frac{x}{1+x^2}$ it follows that

$$g(\pm 1) = \frac{\pm 1}{2}.$$

Consequently, plotting $g(x)$:

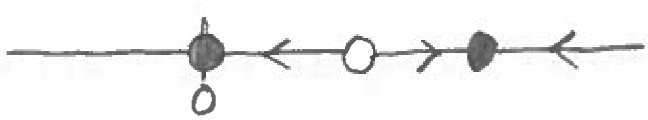


We can deduce the following:

a.) If $0 < r < ½$ then $(*)$ has three fixed points and

$$\lim_{x \to \infty} -rx + \frac{x^2}{1+x^2} = -\infty.$$

Therefore, the phase portrait is given by:



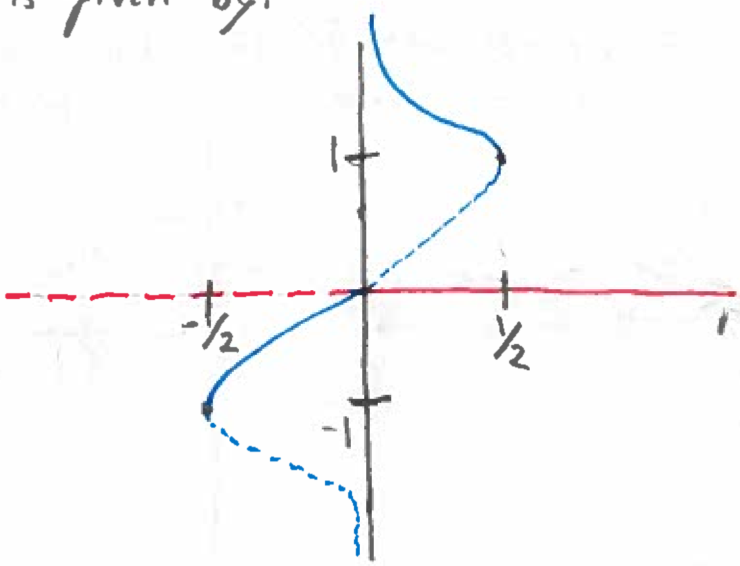
b.) If $-½ < r < 0$ then $(*)$ has three fixed points and

$$\lim_{x \to -\infty} -rx + \frac{x^2}{1+x^2} = \infty.$$

c.) If $r > 1/2$ then (*) has one fixed point

d.) If $r < -1/2$ then (*) hase one fixed point

By items a–d it follows that there is a transcritical bifurcation at $r = 0$ and saddle node bifurcations at $r = \pm 1/2$. The resulting bifurcation is given by:



#3.7.6

Analyze the following model of an epidemic

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - \ell y$$
$$\dot{z} = \ell y.$$

Solution:

First, the population $P = x + y + z$ is constant since:

$$\frac{dP}{dt} = \dot{x} + \dot{y} + \dot{z} = 0.$$

Also,

$$\frac{dx}{dz} = \frac{\dot{x}}{\dot{z}} = -\frac{k}{\ell}x \Rightarrow x(z) = x_0 \exp(-kz/\ell).$$

Consequently,

$$\dot{z} = \ell y = \ell(N - x - z) = \ell(N - z - x_0 \exp(-kz/\ell)).$$

Letting $v = kz/\ell$ it follows that

$$\frac{\ell}{k}\dot{v} = \ell\left(N - \frac{\ell}{k}v - X_0 \exp(-v)\right)$$

$$\Rightarrow \dot{v} = kN - \ell v - kX_0 \exp(-v)$$

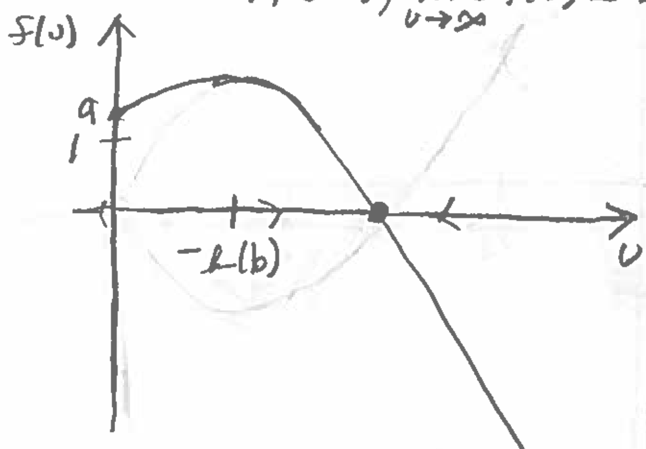
Let $\tau = kX_0 t$:

$$kX_0 \frac{dv}{d\tau} = kN - \ell v - kX_0 \exp(-v)$$

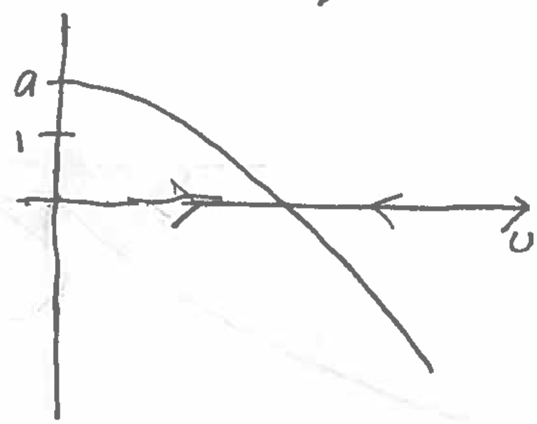
$$\Rightarrow \frac{dv}{d\tau} = \frac{N}{X_0} - \frac{\ell}{kX_0}v - \exp(-v)$$

$$= a - bv - \exp(-v),$$

where $a = N/X_0 > 1$ and $b = \ell/kX_0 > 0$. Let $f(v) = a - bv - \exp(-v)$. Clearly, $f(0) = 0$, $f'(v) = -b + \exp(-v)$, $\lim_{v\to\infty} f(v) = -\infty$. Therefore, plotting $f(v)$:



Case 1: $b < 1$



Case 2: $b > 1$

Clearly, since $\dot{z} = \ell y$ it follows that y obtains it maximum when $\dot{z}$ obtains it maximum. Therefore, an epidemic occurs if $b = \frac{\ell}{kX_0} < 1$. $\blacksquare$

#4.3.6

Analyze the following system:

$$\dot{\theta} = \mu + \sin(\theta) + \cos(2\theta).$$

Solution:

Let $f(\theta) = \sin(\theta) + \cos(2\theta)$. Differentiating it follows that

$$f'(\theta) = \cos(\theta) - 2\sin(2\theta) = \cos(\theta) - 4\sin(\theta)\cos(\theta) = \cos(\theta)(1 - 4\sin(\theta)).$$

Thus the critical points satisfy

$$\theta = \pi/2 + n\pi, \quad \theta = \sin^{-1}(1/4), \pi - \sin^{-1}(1/4)$$

Evaluating at the critical points:

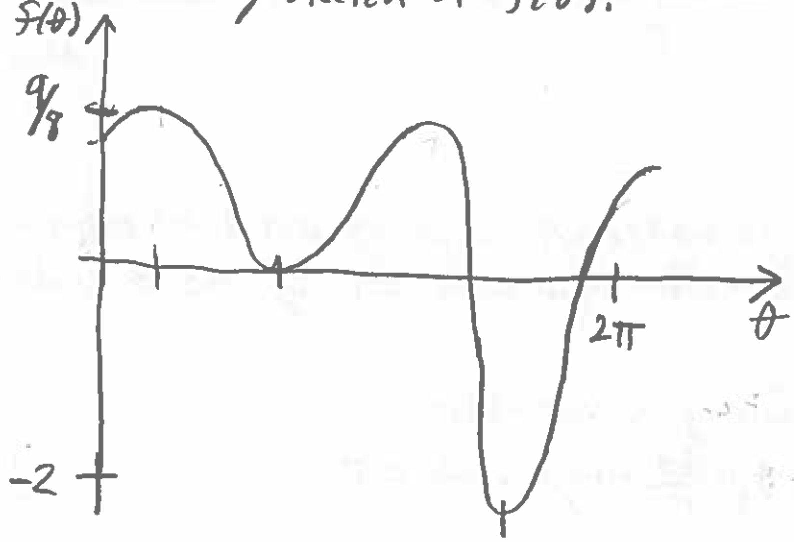
$f(\pi/2) = 0,$

$f(3\pi/2) = -2,$

$$f(\sin^{-1}(1/4)) = 1/4 + \cos^2(\sin^{-1}(1/4)) - \sin^2(\sin^{-1}(1/4))$$
$$= 1/4 + (1 - 1/16) - \frac{1}{16}$$
$$= \frac{5}{4} - \frac{1}{8} = \frac{9}{8}$$

$f(\pi - \sin^{-1}(1/4)) = 9/8,$

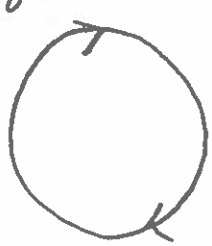
This gives us the following sketch of $f(\theta)$:



Therefore, we have the following cases to consider:

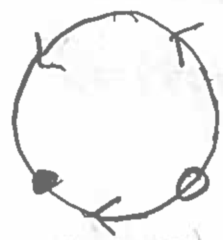
Case 1:

$\nu < -9/8$:



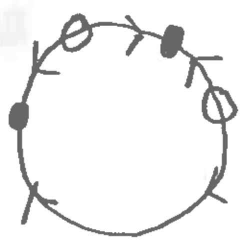
Case 3:

$0 < \nu < -2$



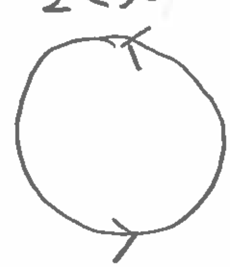
Case 2:

$-9/8 < \nu < 0$

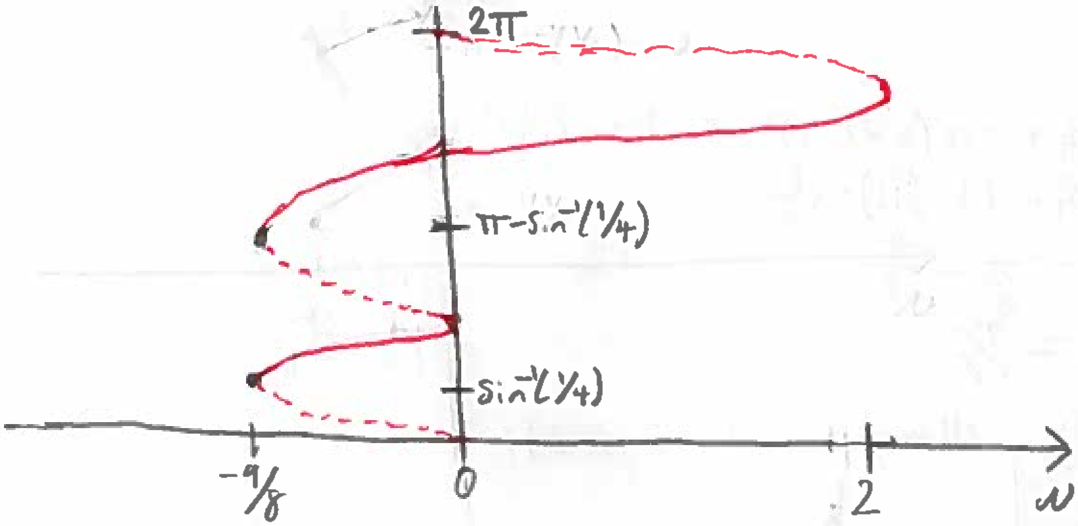


Case 4:

$2 < \nu$



The bifurcation diagram is therefore given by:



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#4.4.1

Find the conditions under which it is valid to approximate the equation $mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) = \Gamma$ by its overdamped limit.

Solution:

Let $\tau = T_{sc}^{-1} t$. Changing variables:

$$\frac{mL^2}{T_{sc}^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{T_{sc}}\frac{d\theta}{d\tau} + mgL\sin(\theta) = \Gamma$$

$$\Rightarrow \frac{mL^2}{mgLT_{sc}^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{mgLT_{sc}}\frac{d\theta}{d\tau} + \sin(\theta) = \gamma$$

Let $T_{sc} = \frac{b}{mgL}$. Therefore,

$$\frac{\cancel{m}L^{\cancel{2}}}{\cancel{m}g\cancel{L}\cdot\frac{b^2}{m^2g^2L^2}}\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} + \sin(\theta) = \gamma$$

$$\Rightarrow \frac{L^3m^2g}{b^2}\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} + \sin(\theta) = \gamma$$

Therefore, the approximation is valid if $\frac{L^3mg}{b^2} << 1$.

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