Homework #4

#5.1.10

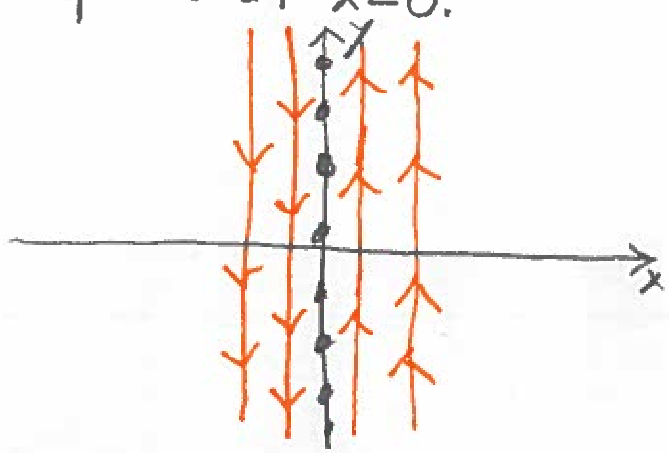
For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

Solution:

a.) $\dot{x}=y$
$\dot{y}=-4x$
The origin is a linear center and hence Liapunov stable.

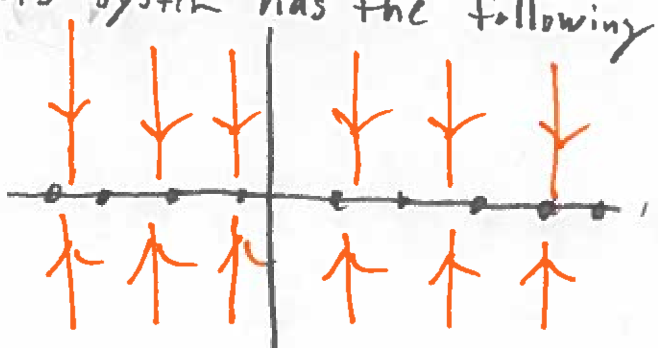
b.) $\dot{x}=2y$
$\dot{y}=x$
The origin is unstable and hence is neither.

c.) $\dot{x}=0,\ \dot{y}=x$
The system has eigenvalues $\lambda_1=\lambda_2=0$ with a line of fixed points at $x=0$:



This system is also neither.

d.) $\dot{x}=0,\ \dot{y}=-y.$
This system has the following phase portrait:



This system is Liapunov stable.

e.) $\dot{x} = -x$, $\dot{y} = -5y$

The eigenvalues are all negative and hence the origin is asymptotically stable.

f.) $\dot{x} = x$, $\dot{y} = y$.

Both eigenvalues > 0 so the system is neither.

#5.3.4

Analyze the following system:

$$\dot{R} = aR + bJ$$
$$\dot{J} = -bR - aJ.$$

Solution:

The eigenvalues satisfy

$$\lambda_1 + \lambda_2 = 0, \quad \lambda_1 \lambda_2 = b^2 - a^2$$
$$\Rightarrow \lambda_1^2 = a^2 - b^2, \quad \lambda_1 = -\lambda_2.$$

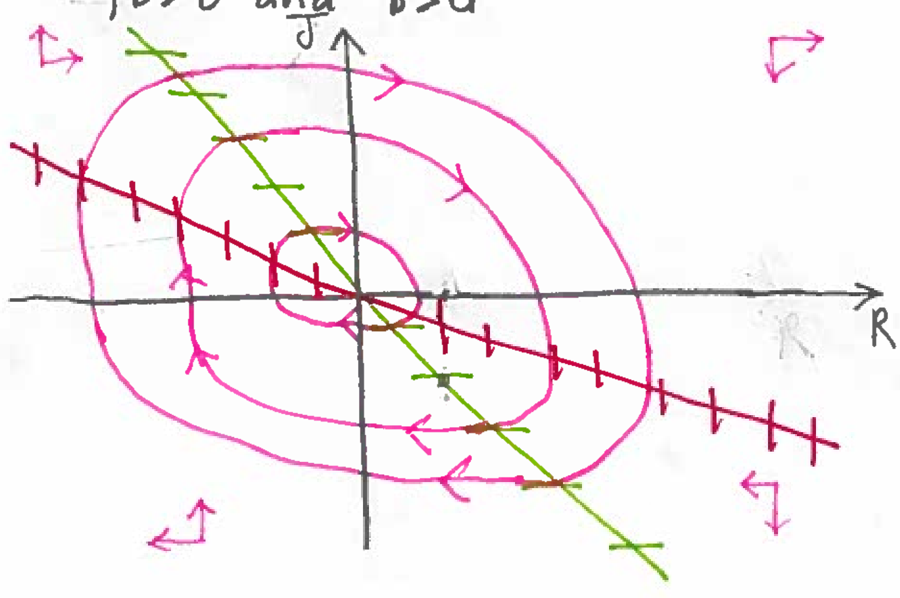
The null-clines are given by:

$\dot{x} = 0$: $J = -\frac{a}{b} R$

$\dot{y} = 0$: $J = -\frac{b}{a} R.$

Case 1:

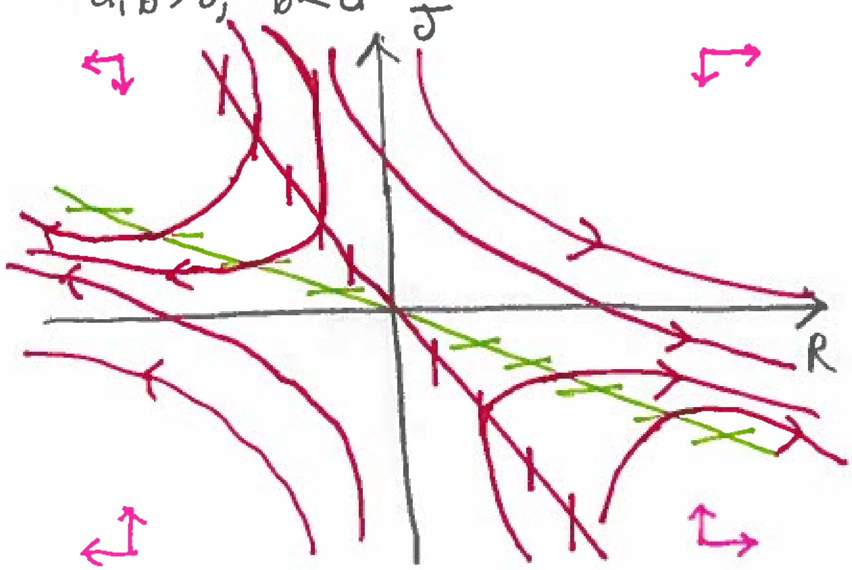
$a, b > 0$ and $b^2 > a^2$



In this case Romeo and Juliet are only in love with each other a quarter of the time. Classic soap opera couple.

Case 2:

$a, b > 0, \; b^2 < a^2$



In this either Juliet or Romeo falls in love with the other but the feelings are not returned.

Case 3:

$a, b < 0$, and $b^2 > a^2$

This case is the same as case 1 with the axis inverted.

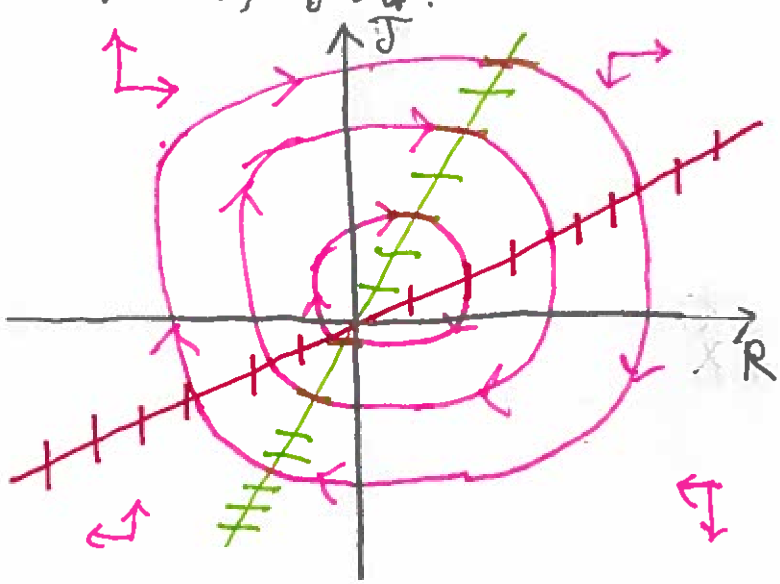
Case 4:

$a, b < 0$ and $b^2 < a^2$,

This case is the same as case 2 with the axis inverted.

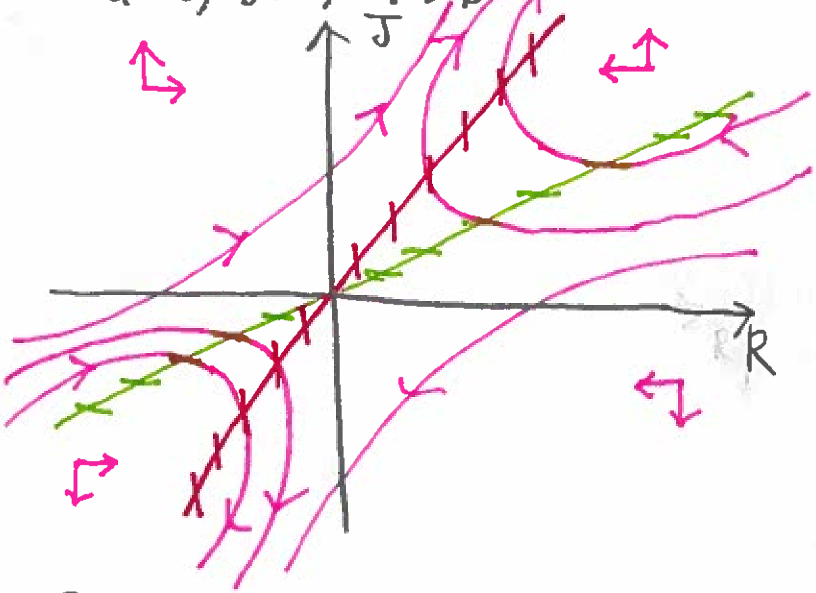
Case 5:

$a < 0, \; b > 0, \; b^2 > a^2$.



Case 6:

$a<0,\ b>0,\ a^2>b^2$



In this case case there is a chance for a happy couple depending on initial conditions!

Case 7:

$b<0,\ a>0,\ b^2>a^2$

This case is the same as case 5 with the axis inverted.

Case 8:

$b<0,\ a>0,\ a^2<b^2$

This case is the same as case 6 with the axis inverted.

#5.2.10

Plot the phase portrait for the system:

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-x-2y\end{aligned} \Rightarrow \dot{\vec{x}}=\begin{pmatrix}0 & 1\\ -1 & -2\end{pmatrix}\vec{x}$$

Solution:

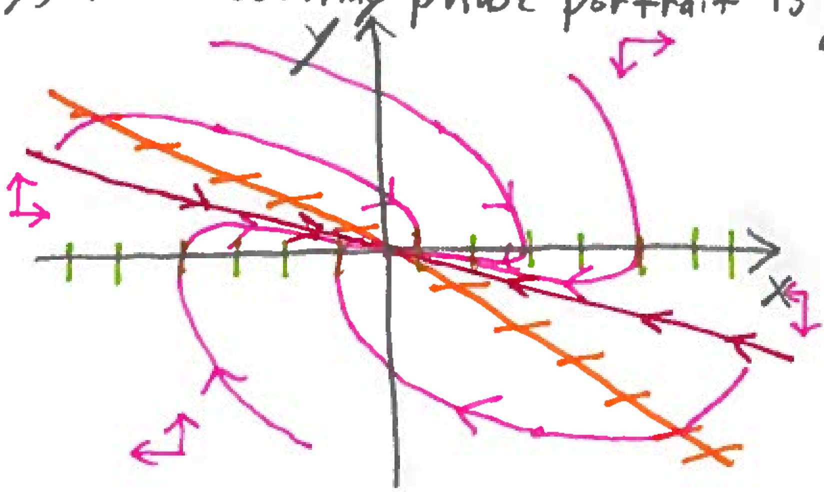
The eigenvalues satisfy

$$\lambda_1+\lambda_2=-2$$
$$\lambda_1\lambda_2=1$$
$$\Rightarrow 1+\lambda_1^2=-2\lambda_1$$
$$\Rightarrow \lambda_1^2+2\lambda_1+1=0$$
$$\Rightarrow \lambda_1=\lambda_2=-1.$$

However there is only a single eigenvector $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
The null-clines are given by

$$\dot{x} = 0 \Rightarrow y = 0$$
$$\dot{y} = 0 \Rightarrow y = -\tfrac{1}{2}x.$$

Consequently, the resulting phase portrait is given by:



#5.2.13

Analyze the following system:

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Solution:

Let $v = \dot{x}$ then

$$\dot{x} = v$$
$$\dot{v} = -\frac{b}{m}v - \frac{k}{m}x$$

The eigenvalues satisfy:

$$\lambda_1 + \lambda_2 = -\frac{b}{m}$$
$$\lambda_1 \lambda_2 = \frac{k}{m}$$

$$\Rightarrow \lambda_1^2 + \frac{b}{m}\lambda_1 + \frac{k}{m} = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

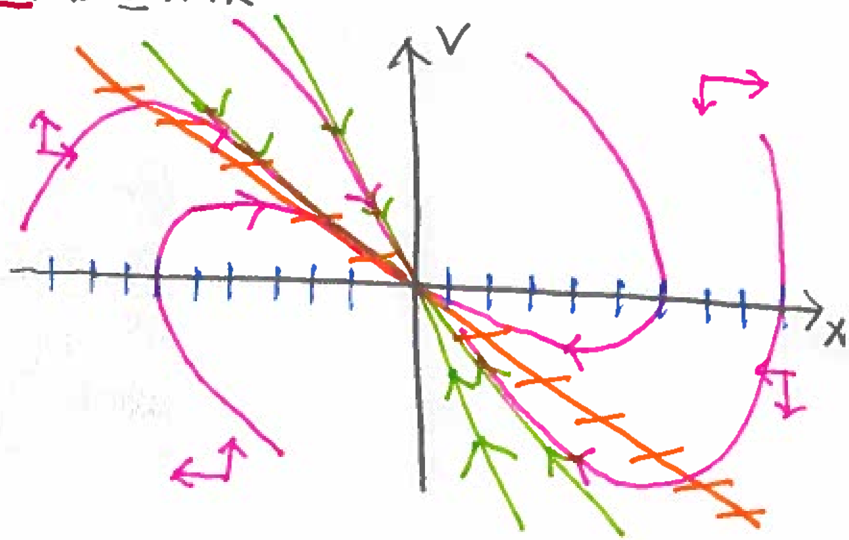
Now since $-b+\sqrt{b^2-4mk} < 0$ if b^2-4mk it follows that $\mathcal{R}(\lambda_{1,2})<0$ for all parameter regimes. There are three cases:

1. $b^2 > 4mk$ Overdamped
2. $b^2 = 4mk$ Critically damped
3. $b^2 < 4mk$ Underdamped vibrations

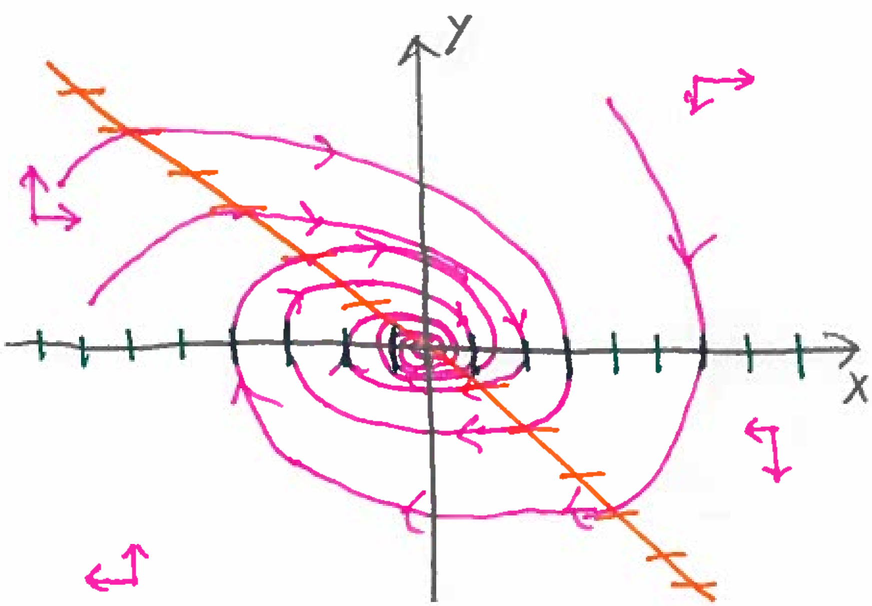
In terms of null-clines we have

N1: $v=0$, $\dot{x}=0$
N2: $v=-\frac{k}{b}x$, $\dot{y}=0$

Case 1: $b^2 \geq 4mk$



Case 2: $b^2 < 4mk$



#6.2.2

Consider the system

$$\dot{x} = y$$
$$\dot{y} = -x + (1-x^2-y^2)y$$

Show that if $|\vec{x}_0| < 1$ then $|\vec{x}(t)| < 1$ for all t.

Solution:

Let $x(t) = \sin(t)$, $y(t) = \cos(t)$. Then,

$$\dot{x}(t) = \cos(t) = y(t)$$
$$\dot{y}(t) = -\sin(t)$$

while

$$-x + (1-x^2-t)y = -\sin(t) + (1-\cos^2(t)-\sin^2(t))\cos(t) = -\sin(t)$$

Hence, $\dot{y} = -x + (1-x^2-y^2)y$. Therefore, the unit circle is a solution and therefore by existence and uniqueness if $|\vec{x}_0| < 1$ its corresponding solution curve $\vec{x}(t)$ must lie inside the unit circle. ∎