Homework #5

#6.3.9

Consider the following system:

$$\dot{x} = y^3 - 4x$$
$$\dot{y} = y^3 - y - 3x$$

Analyze this system.

Solution:

The null-clines are given by:

N1: $x = \frac{1}{4}y^3$

N2: $x = \frac{1}{3}(y^3 - y)$

The fixed points are given by $(0,0)$, $(-2,-2)$, $(2,2)$. The Jacobian is given by:

$$J = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2-1 \end{pmatrix}$$

$$\Rightarrow J(0,0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} \text{ (stable)}$$

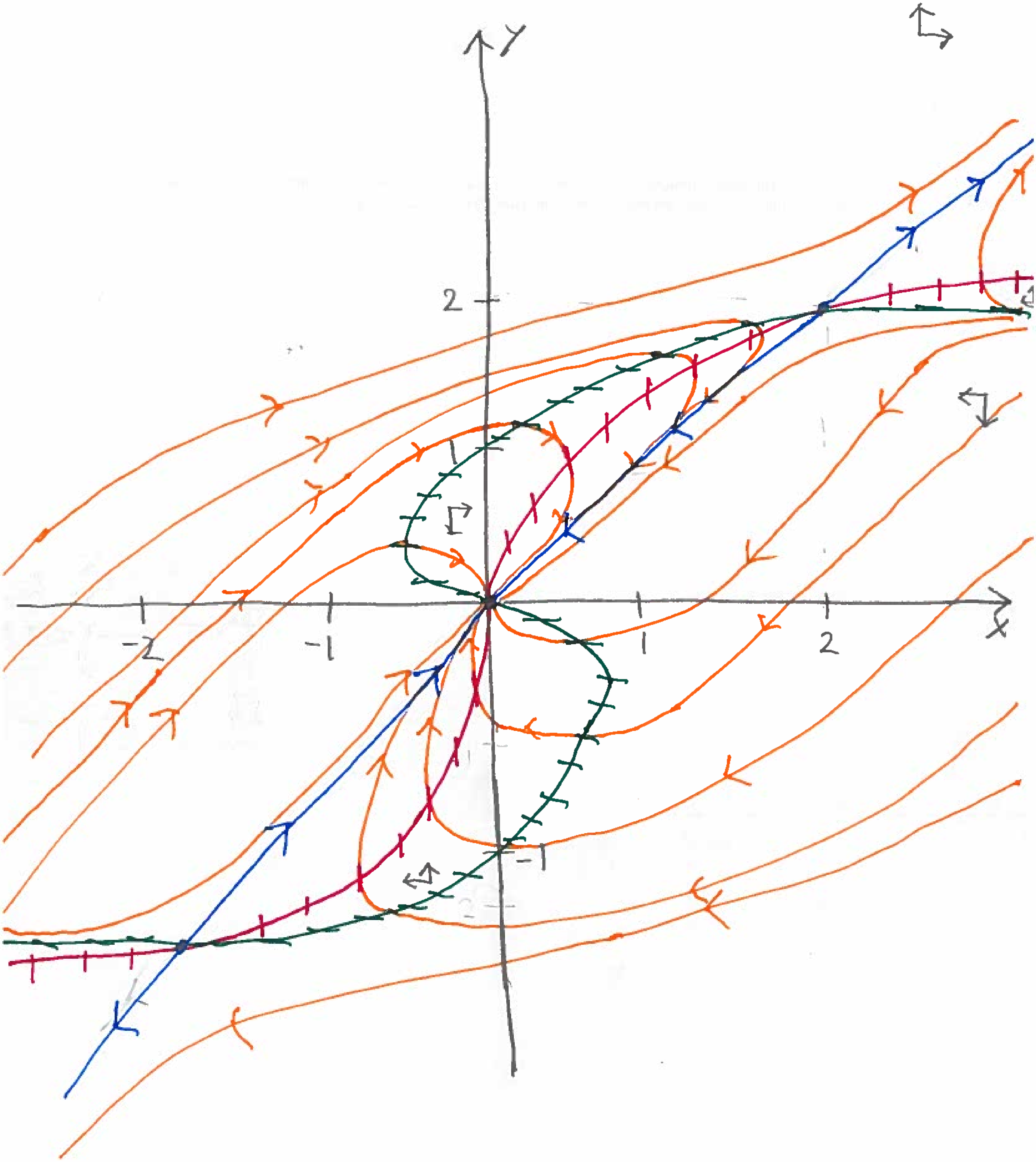
$$J(2,2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \Rightarrow \lambda^2 + 7\lambda - 8 = 0 \Rightarrow \lambda = 8, -1 \text{ (saddle)}$$

$$J(-2,-2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} \text{ (saddle)}$$

If we let $z = |x-y|$ then since $\dot{x} - \dot{y} = y - x$ it follows that

$$\dot{z} = -z$$

Which implies that $z = 0$ is a stable invariant manifold. This information allows us to sketch the phase portrait.



Y
x
2
-2
-1
1
2
-1

#6.3.10

Sketch the phase portrait for the system

$$\dot{x} = xy$$
$$\dot{y} = x^2 - y.$$

Solution:

The null-clines are given by

N1: $x = 0$ $(\dot{x}=0)$

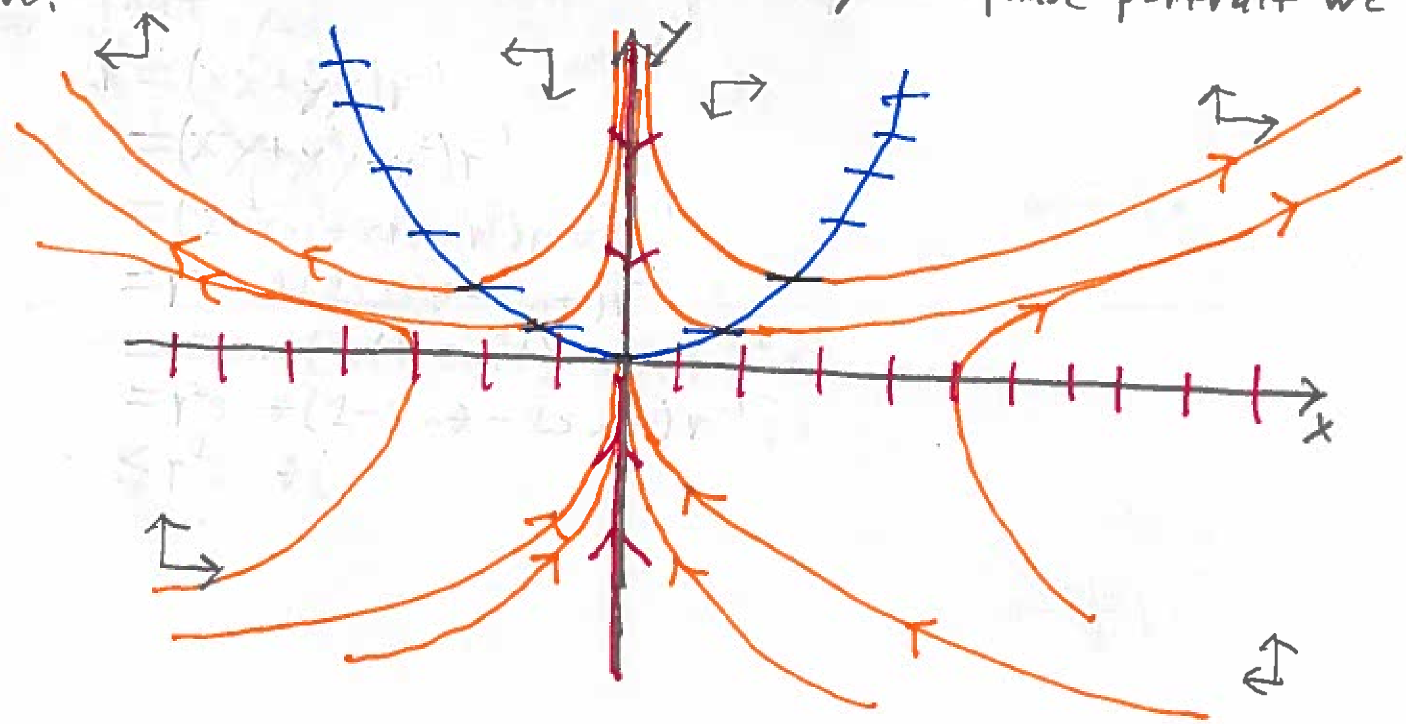
N2: $y = 0$ $(\dot{x}=0)$

N3: $y = x^2$ $(\dot{y}=0)$

The Jacobian is

$$J(x,y) = \begin{pmatrix} y & x \\ 2x & -1 \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which is inconclusive. However sketching the phase portrait we have:



#6.3.13

Analyze the system:

$$\dot{x} = -y - x^3$$
$$\dot{y} = x.$$

Solution:

The null-clines are $y = x^3$, $y = x$. The fixed points are thus $(0,0), (1,1), (-1,-1)$. The Jacobian is given by

$$J(x,y) = \begin{pmatrix} -3x^2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow J(1,1) = \begin{pmatrix} -3 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda^2 + 3\lambda - 1 = 0 \Rightarrow \lambda = \frac{-3 \pm \sqrt{9+4}}{2} \text{ (saddle)}$$

$$J(-1,-1) = \begin{pmatrix} -3 & -1 \\ 1 & 0 \end{pmatrix} \text{ (saddle)}$$

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ (indeterminant).}$$

Converting to polar coordinates we have that

$$\dot{r} = (x \cdot \dot{x} + y\dot{y})r^{-1} = -x^4 r^{-1} < 0$$

$$\dot{\theta} = \frac{d}{dt}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+y^2/x^2} \cdot \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x^2 + y^2 + yx^3}{r^2}$$

$$\Rightarrow \dot{\theta} = 1 + r^2 \sin\theta \cos^3\theta$$

Therefore, if $r < 1$ it follows that solution curves spiral towards the origin.

#6.4.9

Consider the following simple model of a national economy:

$$\dot{I} = I - \alpha C$$
$$\dot{C} = \beta(I - C - G),$$

where $I \geq 0$ is national economy, $C \geq 0$ is rate of consumer spending, and $G \geq 0$ is government spending. Assume $1 < \alpha < \infty$, $0 \leq \beta \leq 1$. Analyze this model in the following cases:

a.) G is constant.

b.) $G = G_0 + kI$, where $k > 0$.

c.) $G = G_0 + kI^2$, where $k > 0$.

Solution:

a.) If $G = G_0$ a constant then, we have the system:

$$\dot{I} = I - \alpha C$$
$$\dot{C} = \beta(I - C - G_0).$$

The null-clines are given by

N1: $C = \frac{1}{\alpha} I$,

N2: $C = I - G_0$,

Hence the fixed point satisfies:

$$\frac{1}{\alpha} I^* = I^* - G_0$$
$$\Rightarrow \frac{1-\alpha}{\alpha} I^* = -G_0$$
$$\Rightarrow I^* = \frac{\alpha}{\alpha-1} G_0, \quad C^* = \frac{1}{\alpha-1} G_0.$$

Let

$$x = I - \frac{\alpha}{\alpha-1} G_0, \quad y = C - \frac{1}{\alpha-1} G_0.$$

$$\Rightarrow \dot{x} = \dot{I} = x + \frac{\alpha}{\alpha-1} G_0 - \alpha y - \frac{\alpha}{\alpha-1} G_0 = x - \alpha y.$$

$$\dot{y} = \dot{C} = \beta\left(x + \frac{\alpha}{\alpha-1} G_0 - y - \frac{1}{\alpha-1} G_0 - G_0\right) = \beta(x - y).$$

That is we obtain the linear system:

$$\dot{x} = x - \alpha y$$
$$\dot{y} = \beta(x-y)$$

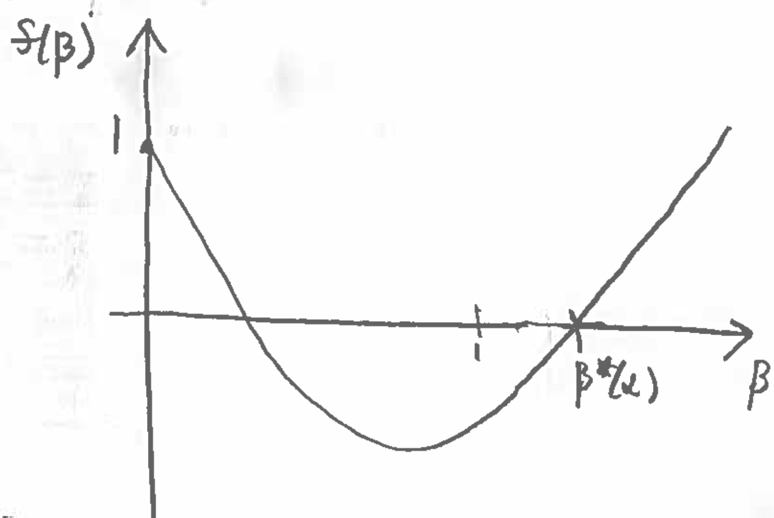
The eigenvalues satisfy

$$\lambda^2 - (1-\beta)\lambda + (-\beta + \alpha\beta) = 0$$

$$\Rightarrow \lambda = \frac{(1-\beta) \pm \sqrt{(1-\beta)^2 - 4\beta(\alpha-1)}}{2} = \frac{-(\beta-1) \pm (\beta-1)\sqrt{1 - \frac{4\beta(\alpha-1)}{(\beta-1)^2}}}{2}$$

Now, we need to investigate the behavior of the polynomial $(1-\beta)^2 - 4\beta(\alpha-1) = f(\beta)$ for $1 \leq \beta < \infty$ Now,

1. $f(1) = -4\beta(\alpha-1) < 0$
2. $\lim_{\beta\to\infty} f(\beta) = \infty$



Therefore, $\exists \beta^*(\alpha)$ such that we have three cases:

1. If $1 < \beta < \beta^*(\alpha)$ then the fixed point is stable spiral
2. If $\beta > \beta^*(\alpha)$ then the fixed point is a stable attractor.
3. If $\beta = 1$ the fixed point is a center.

(So constant goverment spending is stable) b.) Now, if $G = G_0 + kI$, then

$$\dot{I} = I - \alpha C$$
$$\dot{C} = \beta(I - C - G_0 - kI)$$

which is still a linear system so the stability analysis holds over. However, the fixed point now satisfies

$$I^* = \frac{\alpha G_0}{\alpha - k\alpha - 1}, \quad C^* = \frac{G_0}{\alpha - k\alpha - 1}.$$

Hence, if $\alpha - k\alpha - 1 < 0 \Rightarrow k > \frac{\alpha - 1}{\alpha}$ the economy will collapse!! In particular, goverment spending if it is too correlated with income leads to a collapse of the economy.

c.) Now, if $G = G_0 + kI^2$, then

$$\dot{I} = I - \alpha C$$
$$\dot{C} = \beta(I - C - G_0 - kI^2).$$

The null-clines are then

N1: $C = \frac{1}{\alpha} I$

N2: $C = I - G_0 - kI^2 = I(1 - kI) - G_0$

The I coordinate of the fixed point thus satisfies:

$$\frac{1}{\alpha} I = I - G_0 - kI^2$$

$$\Rightarrow 0 = \frac{\alpha - 1}{\alpha} I - kI^2 - G_0$$

Let $f(I) = \frac{\alpha - 1}{\alpha} I - kI^2$

$$\Rightarrow f'(I) = \frac{\alpha - 1}{\alpha} - 2kI$$

Therefore, the maximum value of f is given by:

$$f\left(\frac{\alpha - 1}{2k\alpha}\right) = \frac{(\alpha - 1)^2}{2k\alpha^2} - \frac{k(\alpha - 1)^2}{4k\alpha^2} = \frac{(\alpha - 1)^2}{4k\alpha^2}$$

Consequently, if $G_0 > \frac{(\alpha-1)^2}{4K\alpha^2}$ there are no fixed points.

Now, the fixed points are explicitly given by:

$$I^* = \frac{\frac{\alpha-1}{\alpha} \pm \sqrt{\left(\frac{\alpha-1}{\alpha}\right)^2 - 4KG_0}}{2K},$$

whence if $G_0 > \frac{(\alpha-1)^2}{4K\alpha^2}$ there are two fixed points and if $G_0 < \frac{(\alpha-1)^2}{4K\alpha^2}$ there are none.

#6.4.11

Analyze the following system:

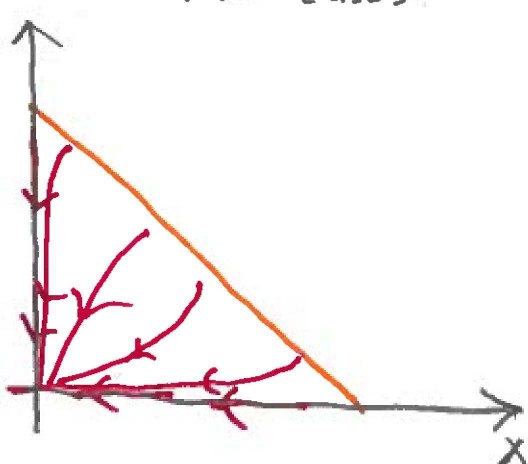
$$\dot{x} = rxz$$
$$\dot{y} = ryz$$
$$\dot{z} = -rxz - ryz$$

Solution:

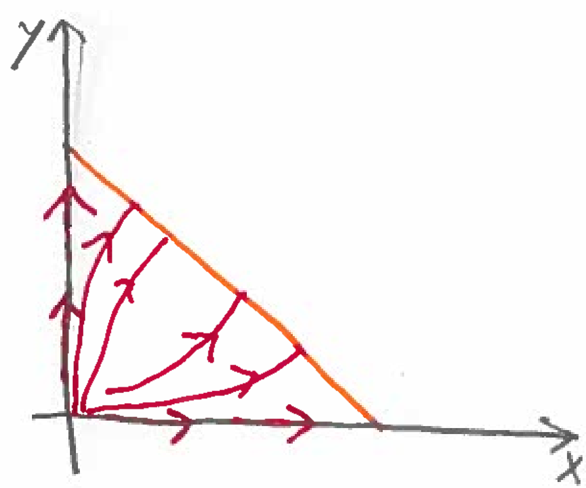
Since $\dot{x} + \dot{z} + \dot{y} = 0$ this system is equivalent to

$$\dot{x} = rx(1-x-y)$$
$$\dot{y} = ry(1-x-y)$$

There are two cases:



$r < 0$



$r > 0$.