Homework #6.

#6.5.12

Analyze the following system:

$$\dot{x} = xy$$
$$\dot{y} = -x^2$$

Solution:

1. Let $E = x^2+y^2$. Along solution curves it follows that

$$\begin{aligned}\frac{dE}{dt} &= 2x\dot{x} + 2y\dot{y} \\ &= 2x^2y - 2x^2y \\ &= 0.\end{aligned}$$

2. In this system the entire line $x=0$ is a line of fixed points.
3. To understand better the behavior of this system we can convert to polar coordinates:

$$r\dot{r} = x\dot{x} + y\dot{y} = 0$$
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{-x^3 - xy^2}{r^2} = \frac{-x(x^2+y^2)}{r^2} = -\cos\theta.$$

Therefore, the phase portrait is given by:

#6.5.14

Consider the following model a glider:

$$\dot{v} = -\sin\theta - Dv^2$$
$$v\dot{\theta} = -\cos\theta + v^2$$

where v is the gliders velocity, θ is the angle with the horizon, and $D>0$ is the drag. Analyze this system.

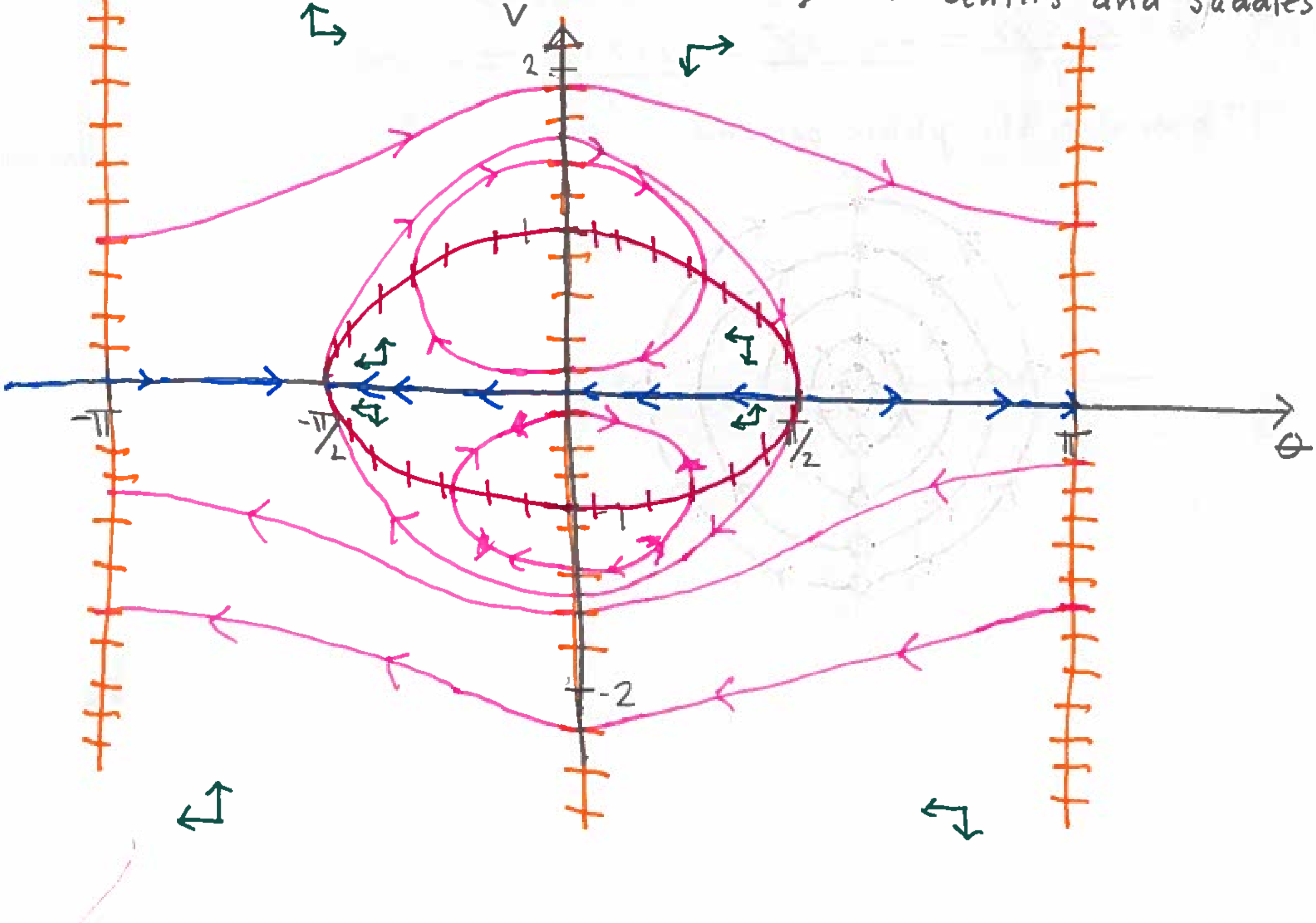
Solution:

1. If $D=0$ consider the following quantity:

$$E = v^3 - 3v\cos\theta$$

$$\Rightarrow \frac{dE}{dt} = 3v^2\dot{v} - 3\dot{v}\cos\theta + 3v\sin\theta\cdot\dot{\theta}$$
$$= 3v^2(-\cancel{\sin\theta} - Dv^2) - 3(\cancel{-\sin\theta} - Dv^2)\cos\theta + 3\cancel{\sin\theta}(-\cancel{\cos\theta} + \cancel{v^2})$$
$$= Dv^2(\cos\theta - 1)$$
$$= 0.$$

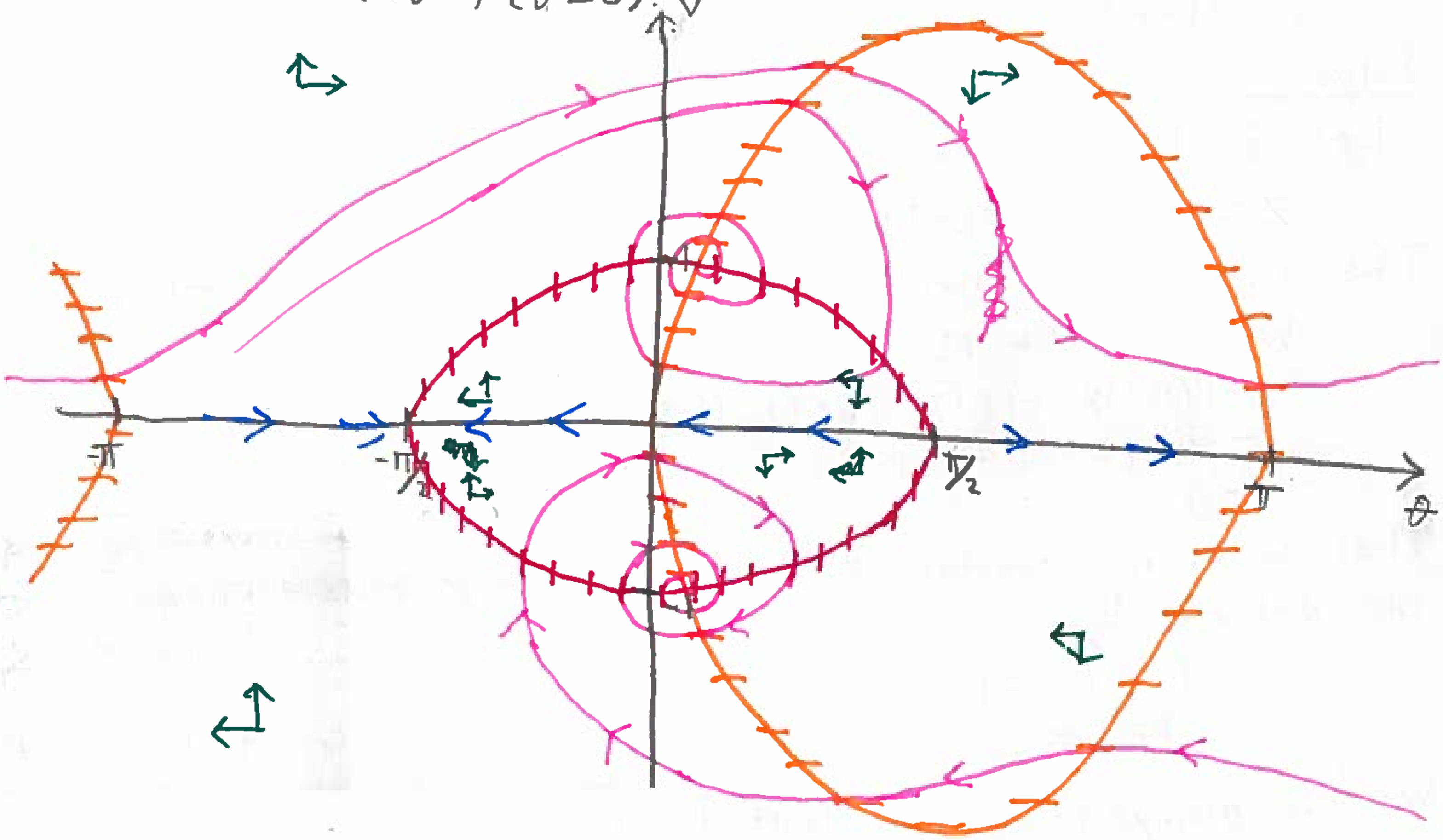
Consequently, for this system we can only have centers and saddles.



2. If $D>0$ we no longer have a conserved quantity. Lets draw the null-clines and analyze this system. The null-clines are given by:

N1: $v = \pm\sqrt{-\frac{1}{D}\sin\theta}$, $(\dot{v}=0)$

N2: $v = \pm\sqrt{\cos\theta}$, $(\dot{\theta}=0)$



In this case the velocity and angle will stabalize which will imply the glider will begin descending.

#6.5.20

Analye the following model for three competing species

$$\dot{P} = P(R-S)$$
$$\dot{R} = R(S-P)$$
$$\dot{S} = S(P-R).$$

Solution:

Let $Z = P+R+S$. Then,

$$\dot{Z} = \dot{P} + \dot{R} + \dot{S} = PR - PS + RS - RP + SP - SR = 0.$$

That is, the total population is constant. Let $W = PRS$. Then,

$$\begin{aligned}\dot{W} &= \dot{P}RS + P\dot{R}S + PR\dot{S} \\ &= P(R-S)RS + PR(S-P)S + PRS(P-R) \\ &= PRS(R-S+S-P+P-R) \\ &= 0.\end{aligned}$$

That is W is a constant as well. Consequently, solution curves are defined by:

$$P+R+S = P_0$$
$$PRS = C_0$$

We can always rescale so that $P_0 = 1$.

$$\Rightarrow P = P_0 - R - S \text{ and } P = \frac{C_0}{RS}$$

$$\Rightarrow P_0 - R - S = \frac{C_0}{RS}$$

$$\Rightarrow P_0 RS - R^2 S - RS^2 = C_0$$

$$\Rightarrow RS^2 + R(R-P_0)S + C_0 = 0$$

$$\Rightarrow S = \frac{(1-R)R \pm \sqrt{R^2(R-P_0)^2 - 4RC_0}}{2R}$$

Thus, we get closed curves for sufficiently small values of C_0. ∎

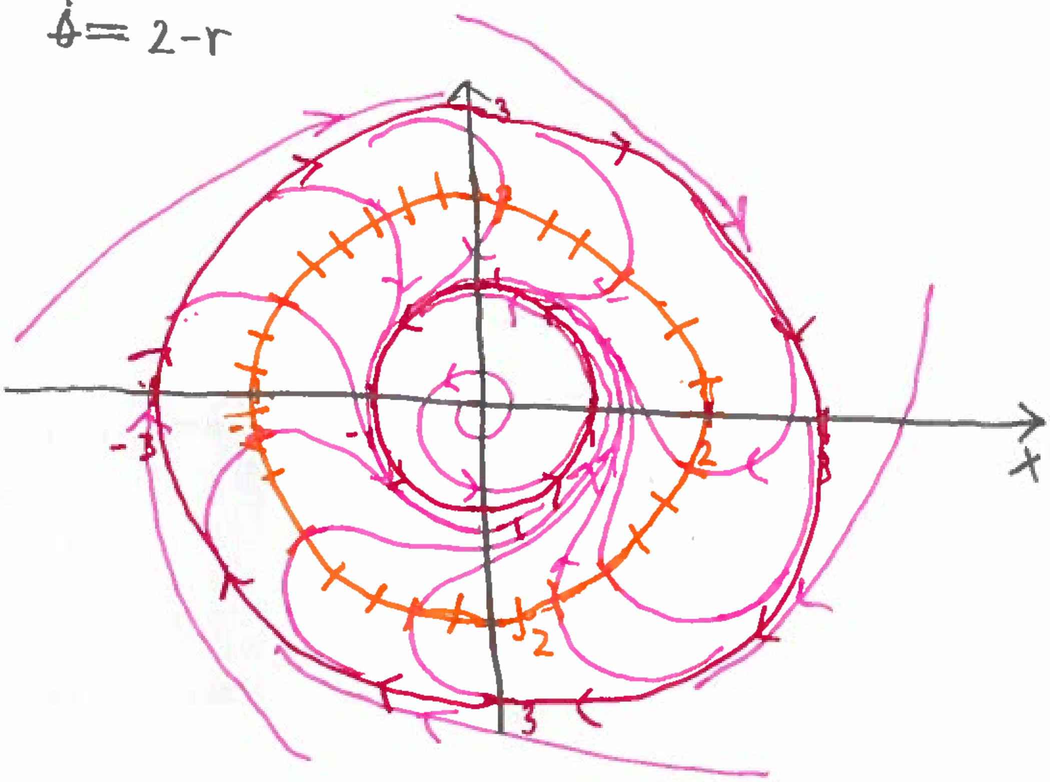
#6.8.9

Construct a system with two closed trajectories, one containing the other, in which one runs clockwise the other counterclockwise and there is only one fixed point.

Solution:

In polar coordinates consider the following system:

$$\dot{r} = r(1-r)(3-r)$$
$$\dot{\theta} = 2-r$$



#7.2.12

Show that

$$\dot{x} = -x + 2y^3 - 2y^4$$
$$\dot{y} = -x - y + xy$$

has no periodic solutions.

Solution:

Let $L = x^m + ay^n$. Then,

$$\dot{L} = mx^{m-1}\cdot\dot{x} + any^{n-1}\cdot\dot{y}$$
$$= m(-x^m + 2y^3x^{m-1} - 2y^4\cdot x^{m-1}) + an(-xy^{n-1} - y^n + xy^n)$$

If $m=2$ then

$$\dot{L} = -2x^2 + 4y^3x - 4y^4x + an(-xy^{n-1} - y^n + xy^n)$$

Matching powers we need $n=4$ and $a=1$. Therefore,

$$\dot{L} = -2x^2 + 4y^3x - 4y^4x - 4xy^3 - 4y^4 + 4xy^4$$

$$= -2x^2 - 4y^4$$

Therefore, $L = x^2 + y^4$ is a Lyapunov function. ∎