Homework #7

#7.2.14

Analyze the following system:

$$\dot{x} = x^2 - y - 1$$
$$\dot{y} = y(x-2).$$

Solution:

The null-clines are given by:

$y = x^2 - 1 \quad (\dot{x} = 0)$

$y = 0 \quad (\dot{y} = 0)$

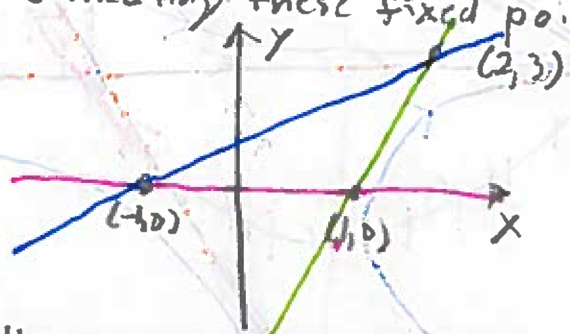
$x = 2 \quad (\dot{y} = 0)$

Consequently, the fixed points are given by

$(-1, 0), (1, 0), (2, 3).$

Consider the three lines connecting these fixed points:

1. $y_1 = 0$
2. $y_2 = x + 1$
3. $y_3 = 3x - 3$



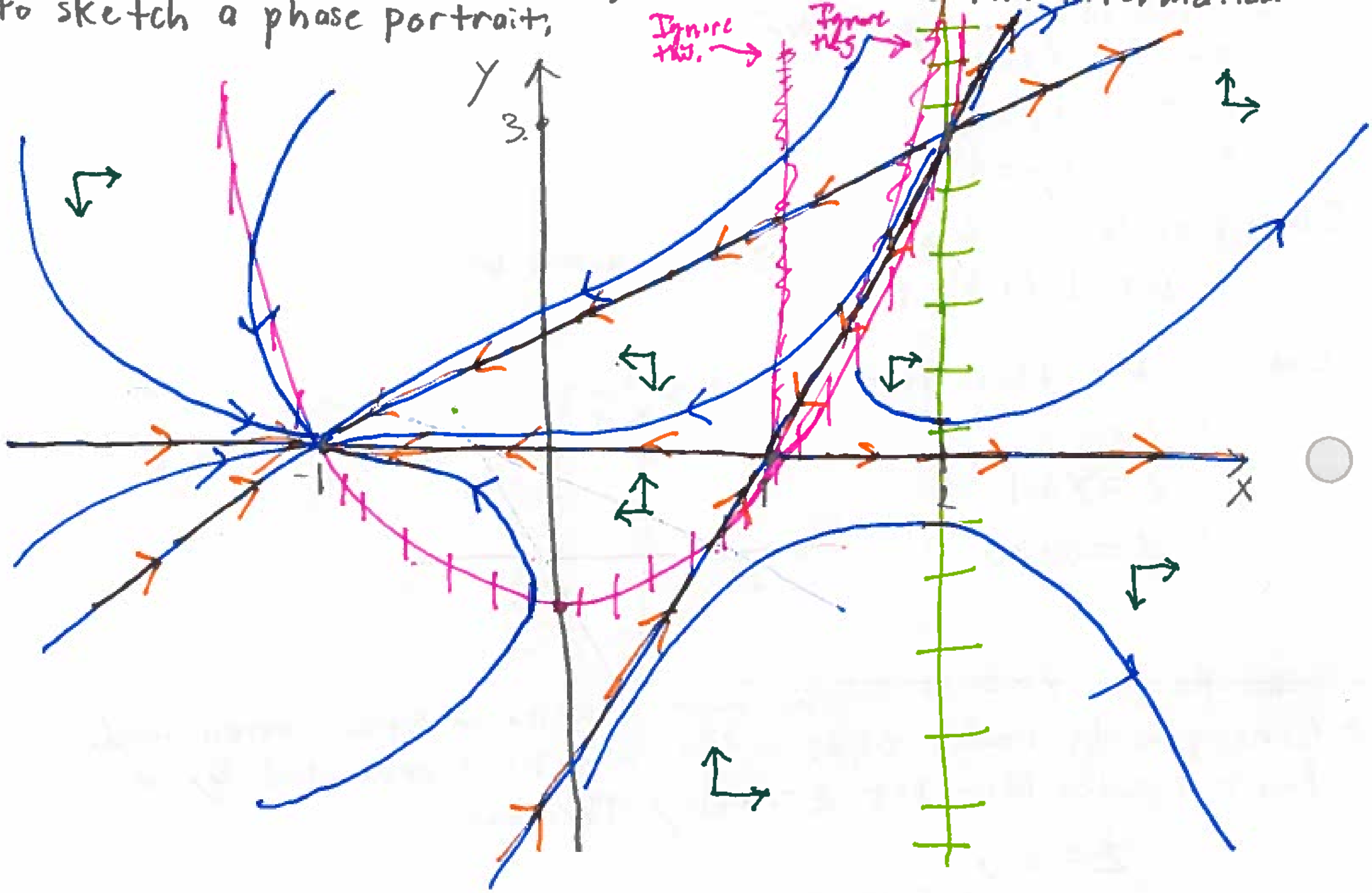
a. Since on y_1, $\dot{y} = 0$ it follows that y_1 is an invariant manifold.
b. Consequently, neither $(-1, 0)$ or $(1, 0)$ can be surrounded by a limit cycle. Now, let $z = x + 1 - y$. Therefore,

$$\dot{z} = \dot{x} - \dot{y}$$
$$= x^2 - y - 1 - yx + 2y$$
$$= x^2 + y - yx - 1$$
$$= \cancel{x^2} + (x + \cancel{1} - z)y - x(\cancel{x} + \cancel{1} - z) - \cancel{1}$$
$$= -z - xz.$$

Therefore, y_2 is an invariant manifold. Consequently $(2, 3)$ cannot be surrounded by a limit cycle. Now, let $w = y - 3x + 3$. Therefore,

$$\dot{w} = \dot{y} - 3\dot{x}$$
$$= y(x-2) - 3(x^2 - y - 1)$$
$$= (w+3x-3)(x-2) - 3(x^2 - w - 3x + 3 - 1)$$
$$= wx - 2w + \cancel{3x^2} - \cancel{6x} - \cancel{3x} + \cancel{6} - \cancel{3x^2} - 3w + \cancel{9x} - \cancel{9} + \cancel{3}$$
$$= wx - 5w,$$

Which vanishes if $w=0$. Therefore, y_3 is an invariant manifold. Consequently (2, 3) cannot contain a limit cycle. We can use this information to sketch a phase portrait,



#7.3.4

Analyze the following system:

$$\dot{x} = x(1-4x^2-y^2) - \tfrac{1}{2}y(1+x)$$
$$\dot{y} = y(1-4x^2-y^2) + 2x(1+x).$$

Solution:

$$J(0,0) = \begin{pmatrix} 1 & -1/2 \\ 2 & 1 \end{pmatrix} \Rightarrow \lambda^2 - 2\lambda + 1 = 0.$$

The eigenvalues are thus given by $\lambda = 1$ which implies the origin is unstable. Now, define $V = (1-4x^2-y^2)^2$. Therefore,

$$\begin{aligned}
\dot{V} &= 2\cdot V^{1/2}\cdot(0 - 8x\cdot\dot{x} - 2y\dot{y}) \\
&= 2V^{1/2}\cdot(0 - 8x^2\cdot V^{1/2} + 4xy(1+x) - 2y^2\cdot V^{1/2} + 4xy(1+x)) \\
&= 2V^{1/2}(1 - 8x^2V^{1/2} - 2y^2V^{1/2}) \\
&= 4V((1-4x^2-y^2-1) \\
&= 4V(V^{1/2}-1)
\end{aligned}$$

For this 1-D system it follows that $V \to 1$ as $t \to \infty$.

#7.3.6

Consider the oscillator equation $\ddot{x} + F(x,\dot{x})\dot{x} + x = 0$, where $F(x,\dot{x}) < 0$ if $r < a$ and $F(x,\dot{x}) > 0$ if $r > b$. Show this system has a limit cycle.

Solution:

Let $r^2 = x^2 + \dot{x}^2$, $\theta = \tan^{-1}\left(\frac{\dot{x}}{x}\right)$. Then,

$$\begin{aligned}
\dot{r} &= (x\cdot\dot{x} + \dot{x}\cdot\ddot{x})/r \\
&= r\cos\theta\sin\theta + \sin\theta(-x - F(x,\dot{x})\dot{x}) \\
&= r\cos\theta\sin\theta - r\sin\theta\cos\theta - r\sin\theta F(x,\dot{x}) \\
&= -r\sin\theta F(x,\dot{x}).
\end{aligned}$$

Consequently, the annulus $a \le r \le b$ is a trapping region with no fixed points. Hence, it must contain a limit cycle.

#7.3.11

Analyze the following system

$$\dot{r} = r(1-r^2)[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2]$$
$$\dot{\theta} = r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2$$

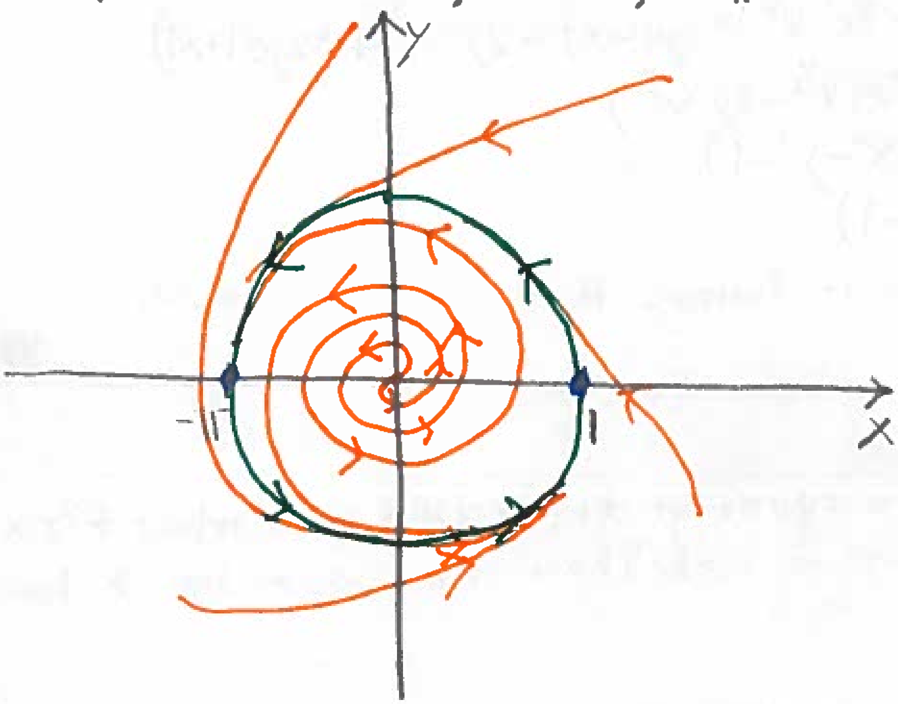
Solution;

Since $\dot{\theta} > 0$ and is the sum of two positive terms it follows that $\dot{\theta} = 0$ if and only if

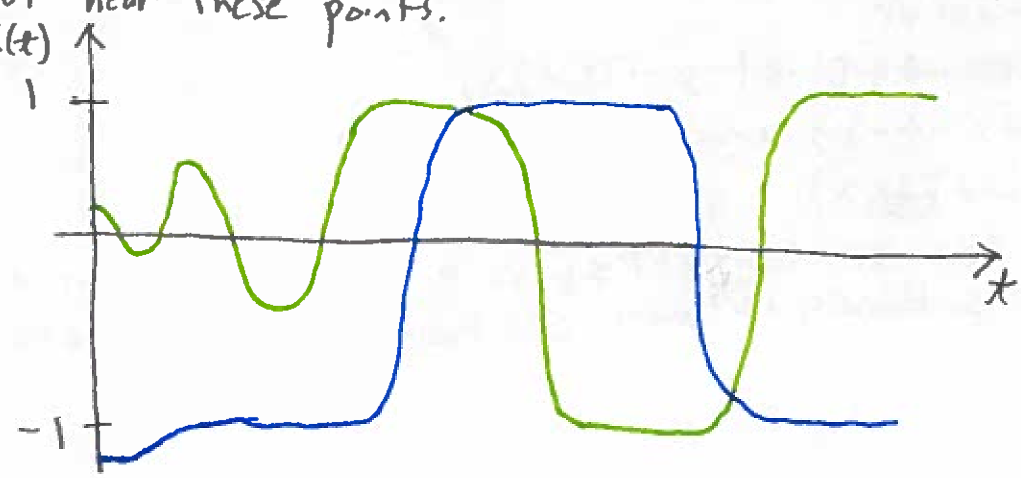
$$r^2\sin^2\theta = 0 \quad \text{and} \quad r^2\cos^2\theta = 1$$

$$\Rightarrow \theta = n\pi \text{ and } r = 1.$$

The fixed points are clearly $\theta = n\pi$, $r = 1$.



Since $\dot{X} \approx 0$ near $x = \pm 1$ it follow that the the trajectory will flatten out near these points.



#7.3.12

Analyze the following rock, paper, scissors system:

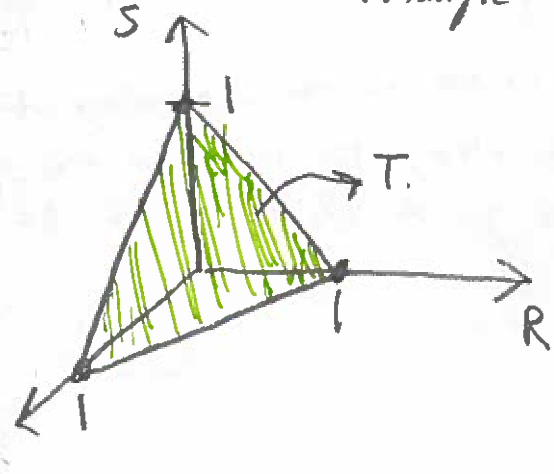
$$\dot{P} = P[(aR-S)-(a-1)(PR+RS+PS)]$$
$$\dot{R} = R[(aS-P)-(a-1)(PR+RS+PS)]$$
$$\dot{S} = S[(aP-R)-(a-1)(PR+RS+PS)].$$

Solution:

Let $E_1 = P+R+S$. Then,

$$\dot{E}_1 = -(a-1)E_1(PR+RS+PS)+a(PR+RS+SP)-PS-RP-SR$$
$$=(1-E_1)(a-1)(PR+RS+PS).$$

Consequently, if $E_1=1$, then $\dot{E}_1=0$. Since $P,R,S>0$ it follows that the dynamics can be restricted to the plane $P+R+S=1$ in the first quadrant which is a triangle T:



Now, consider the lines connecting the fixed points

1. ℓ_1: $P+R=1$
2. ℓ_2: $P+S=1$
3. ℓ_3: $R+S=1$

Let $z_1 = 1-P-R$

$$\Rightarrow \dot{z}_1 = 0-\dot{P}-\dot{R}$$
$$=0-a(PR+RS)+PS+PR+(a-1)(PR+RS+PS)(P+R)$$

If $S=0$ we have:

$$\dot{z}_1 = 0-(a+1)PR+(a-1)PR(P+R) = -PR(a-1)z_1$$

Therefore, ℓ_1 is an invariant manifold. Similar arguments show that ℓ_2, ℓ_3 are invariant. Now, define $E_2 = PRS$. Calculating it follows that:

$$\dot{E}_2 = \dot{P}RS + P\dot{R}S + PR\dot{S}$$
$$= E_2 \cdot [a(R+S+P) - (S+P+R) - 3(a-1)(PR+RS+PS)]$$

Therefore, on T:

$$\dot{E}_2 = E_2(a-1)(1-3(PR+RS+PS))$$
$$= E_2(a-1)((P+R+S)^2 - 3PR - 3RS - 3PS)$$
$$= E_2(a-1)(P^2 + 2PR + 2PS + R^2 + 2RS + S^2 - 3PR - 3RS - 3PS)$$
$$= \frac{E_2(a-1)}{2}(P^2 - 2PR + R^2 + P^2 - 2PS + S^2 + R^2 - 2RS + S^2)$$
$$= \frac{E_2(a-1)}{2}\left[(P-R)^2 + (P-S)^2 + (R-S)^2\right].$$

Consequently, E_2 is monotone except on the boundary of T and at the point $(1/3, 1/3, 1/3)$. Now, clearly E_2 is maximized at $(1/3, 1/3, 1/3)$. Therefore, if $a > 1$ all solutions go to $(1/3, 1/3, 1/3)$ if $a < 1$ we go to the boundary of T_1.