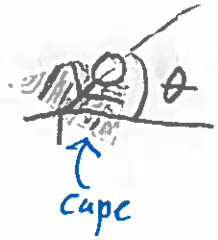
Lecture Notes for Sarah (Modeling superman)

A glider flying at speed v with an angle θ with the horizon is modeled by:

$$\dot{v} = \underbrace{-\sin\theta}_{\text{gravity}} \underbrace{- Dv^2}_{\text{drag}}, \qquad *$$

$$v\dot{\theta} = \underbrace{-\cos\theta}_{\text{gravity}} + \underbrace{v^2}_{\text{lift.}}$$



If $(v(t), \theta(t))$ solve $*$ then the Cartesian coordinates of the glider can be recovered:

$$\dot{x} = v(t)\cos(\theta(t)) \longrightarrow x(t) = x_0 + \int_{t_0}^{t} v(s)\cos(\theta(s))\,ds,$$

$$\dot{y} = v(t)\sin(\theta(t)) \longrightarrow y(t) = y_0 + \int_{t_0}^{t} v(s)\sin(\theta(s))\,ds.$$

Case 1:

Lets suppose $D=0$. Define $E: \mathbb{R}\times S^1 \to \mathbb{R}$ by

$$E(v,\theta) = v^3 - 3v\cos\theta$$

Then along a solution curve:

$$\begin{aligned}\frac{dE}{dt} &= 3v^2\dot{v} - 3\cos\theta\,\dot{v} + 3v\sin\theta\,\dot{\theta}\\ &= 3v^2(\cancel{-\sin\theta} - Dv^2) - 3\cos\theta(\cancel{-\sin\theta} - Dv^2) + 3v\sin\theta\left(\cancel{-\frac{\cos\theta}{v}} + v\right)\\ &= 3v^2 D(\cos\theta - v^2)\\ &= 0 \quad (\text{if } D=0)\end{aligned}$$

Now for this system we have the following null-clines:

N1: $\theta = k\pi$ $(\dot{v}=0)$

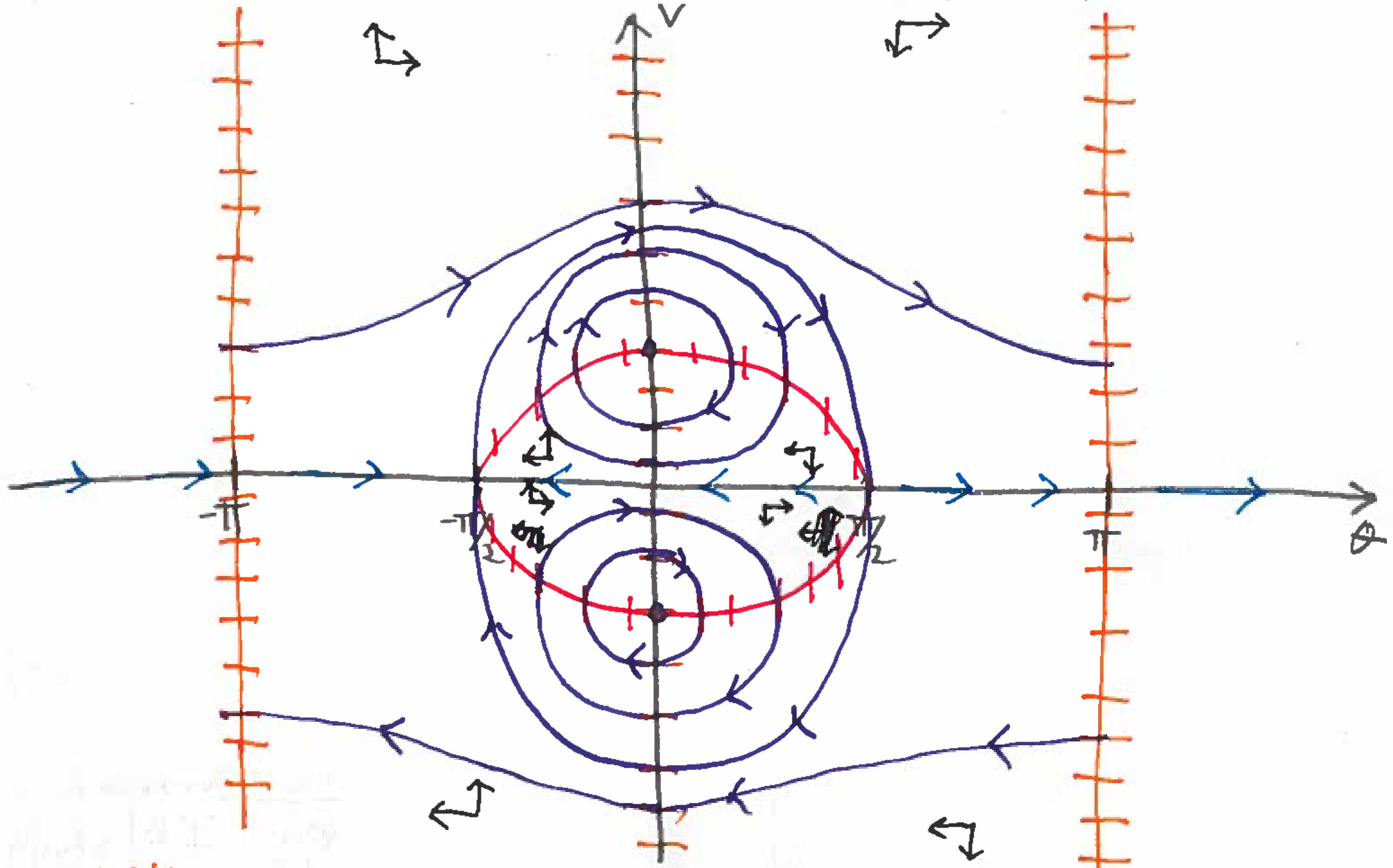
N2: $v = \pm\sqrt{\cos\theta}$ $(\dot{\theta}=0)$.

Also, on the curve $v=0$ we have that:

$$\frac{dv}{d\theta} = \frac{\dot{v}}{\dot{\theta}} = \lim_{v\to 0} \frac{v(-\sin\theta - Dv^2)}{-\cos\theta + v^2} = 0.$$

That is, although $\dot{\theta}$ is undefined when $v=0$, nearby the flow is aligned in the θ direction.

We can use this information to sketch a phase portrait:

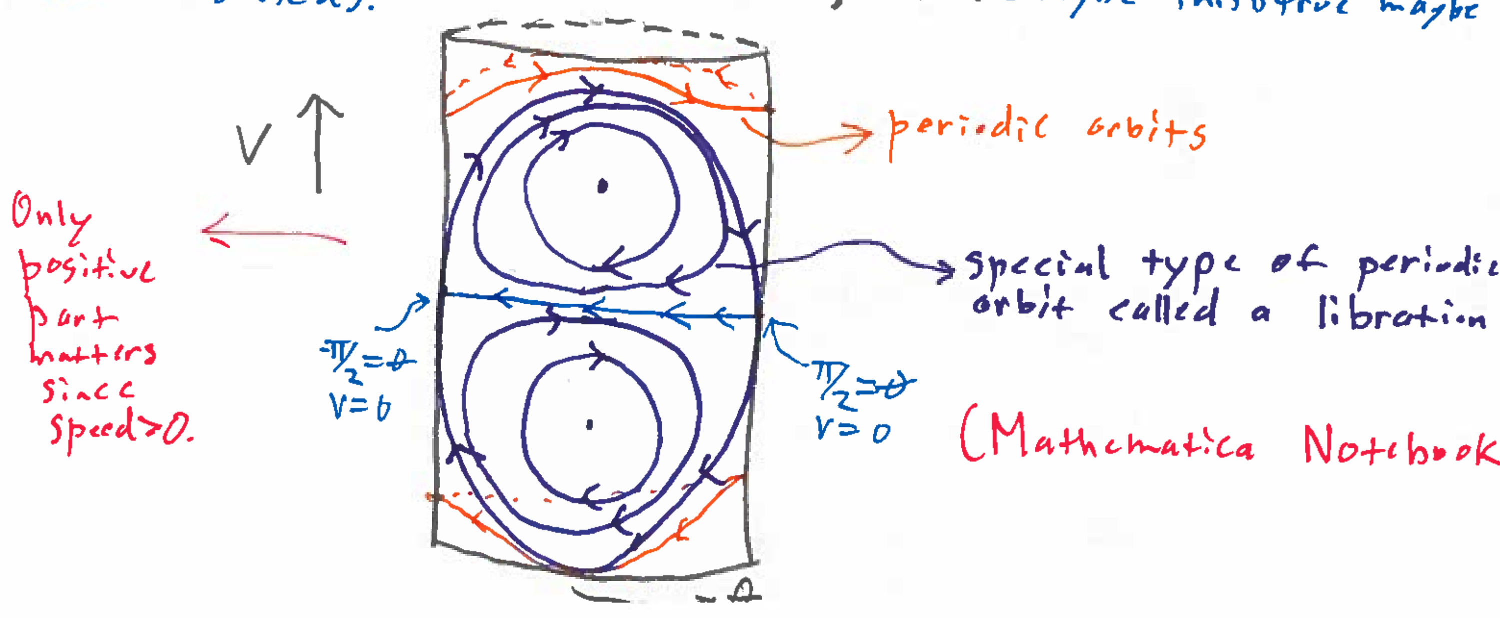


* N1
* N2
* V=0
* Solution Curves

Important!!

We can only conclude that there are nonlinear centers since the system is conservative. Moreover, no eigenvalue analysis is necessary. This is the point of conservative systems!!

The real phase space is $\mathbb{R} \times S^1$! It is more illuminating to plot the solution curves on a cylinder. (Maybe this is true maybe not. I am biased).



(Mathematica Notebook).

Case 2:

If $D>0$ we no longer have a conserved quantity. Lets solve for the fixed points analytically!

$$v^2 = -\frac{\sin\theta}{D}$$

$$v^2 = \cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = -D$$

$$\Rightarrow \theta = \tan^{-1}(-D)$$

$$\Rightarrow v^2 = -\frac{\sin(\tan^{-1}(-D))}{D}$$

$$= \frac{1}{(1+D^2)^{1/2}}$$

Therefore, our fixed point is simply

$$v^* = \frac{1}{(1+D^2)^{1/4}}$$

$$\theta^* = \tan^{-1}(-D)$$

Analysis can show that this fixed point is an attractor. Consequently, the glider becomes locked at the angle $\theta^* = \tan^{-1}(-D)$ and begins diving at speed $v^* = \frac{1}{(1+D^2)^{1/4}}$.

Lecture Notes for Sarah (Rock Paper Scissors)

$$\dot{P} = P(R-S), \quad P>0 \text{ population of paper}$$
$$\dot{R} = R(S-P), \quad R>0 \text{ population of rocks}$$
$$\dot{S} = S(P-R) \quad S>0 \text{ Population of scissors.}$$

Let $T = P+R+S$ denote the total population. Then,

$$\dot{T} = \dot{P} + \dot{R} + \dot{S}$$
$$= PR - PS + RS - RP + SP - SR$$
$$= 0$$

Therefore, total population is conserved. Without loss of generality assume the $T=1$.

Let $E = PRS$. Then,

$$\dot{E} = \dot{P}RS + P\dot{R}S + PR\dot{S}$$
$$= P(R-S)RS + PR(S-P)S + PRS(P-R)$$
$$= PRS(R-S+S-P+P-R)$$
$$= 0.$$

Therefore, if we let $S = \{(P,R,S) \in \mathbb{R}^3 : P+R+S=1\}$ and $S_c = \{(P,R,S) \in \mathbb{R}^3 : PRS = C\}$ the solution curves to this system in $\mathbb{R}^3$ lie in the set $S_1 \cap S_c$.

However, we can also do this analytically. Since $\dot{T}=0$ it follows that:

$$\dot{P} = P(2R+P-1)$$
$$\dot{R} = R(1-R-2P)$$

The reduced system also has a conserved quantity

$$E = (1-P-R)PR,$$

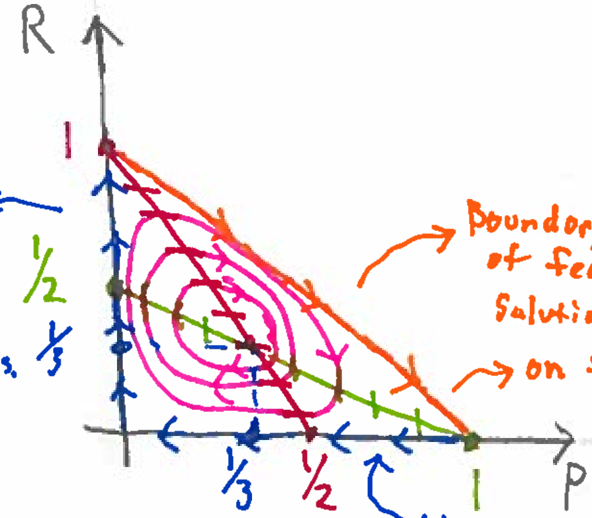
which implies the system can only have centers and saddles. The null-clines for this system are given by:

N1: $P=0$, $(\dot{P}=0)$, N3: $R = \frac{1}{2} - \frac{1}{2}P$ $(\dot{P}=0)$

N2: $R=0$, $(\dot{R}=0)$, N4: $R = 1-2P$ $(\dot{R}=0)$

Phase portrait:

No paper rocks dominate, eventually no scissors.



Boundary of feasible solutions.

on this manifold there are no scissors so rock dies out to paper.

No rocks paper dies, also only scissors left.

If we let $Z = 1 - R - P$ then

$$\dot{Z} = -\dot{R} - \dot{P} = -2PR - P^2 + P - R + R^2 + 2PR$$

$$\Rightarrow \dot{Z} = P - R + R^2 - P^2$$

$$= P - R + (R-P)(R+P)$$

$$= (P-R)(1-R-P)$$

$$= (P-R)Z$$

Therefore, the boundary curve $R = 1 - P$ is an invariant manifold.

*The above example is known as a cycle graph.