Analysis
Spring 2018
Exam 1
02/16/18

Name (Print): Key

This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

   (a) (25 points) Let (X, d) be a metric space and let x_n be a Cauchy sequence in X with respect to this metric. Prove that x_n is bounded. **Note:** You cannot assume that (X, d) is a complete metric space.

Let x_n be a Cauchy sequence in (X,d). Therefore, there exists $N \in \mathbb{N}$ such that $m,n \geq N \Rightarrow d(x_n, x_m) < 1$. Therefore, for all $m \geq N$ it follows that

$$d(x_m, x_N) < 1.$$

Consequently, for all $n \in \mathbb{N}$:

$$d(x_n, x_N) \leq \max\left\{\max_{1 \leq i \leq N} d(x_i, x_N), 1\right\}.$$

(b) (25 points) Let x_n be a Cauchy sequence in a metric space (X, d). Suppose there exists a subsequence x_{n_k} that converges to $x \in X$ with respect to this metric. Prove that x_n converges to x.

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x).$$

Since x_n is Cauchy and $x_{n_k} \to x$ it follows that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $k, n \geq N$ implies

$$d(x_n, x_{n_k}) < \varepsilon/2 \text{ and } d(x_{n_k}, x) < \varepsilon/2.$$

2. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

(a) (25 points) Let (X, d) be a metric space. Prove that $(X, \sqrt{d})$ is a metric space.

Let $f(x) = \sqrt{x}$. Since $f(0) = 0$, f is increasing, and f is concave down it follows that

$$\sqrt{d} = f(d)$$

is a metric.

(b) (25 points) Let X be a vector space with the scalar field $\mathbb{R}$, and let d be a metric on X satisfying $d(x,y) = d(x-y,0)$ and $d(\alpha x, \alpha y) = |\alpha| d(x,y)$ for every $x, y \in X$ and every $\alpha \in \mathbb{R}$. Prove that $\|x\| = d(x,0)$ defines a norm on X.

1. Suppose $\|x\|=0$. Then,

$$d(x,0)=0$$
$$\Rightarrow x=0$$

If $x=0$ then $\|x\|=d(x,0)=0$.

2. $\|x\|=d(x,0)\geq 0$.

3. $\|\alpha x\| = d(\alpha x, 0) = |\alpha| d(x,0) = |\alpha|\cdot\|x\|$.

4. $$\begin{aligned}\|x+y\| &= d(x+y,0)\\ &= d(x,-y)\\ &\leq d(x,0)+d(0,-y)\\ &= d(x,0)+d(-y,0)\\ &= d(x,0)+d(y,0)\\ &= \|x\|+\|y\|.\end{aligned}$$

3. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

   (a) (25 points) Let (X, d) be a metric space, and let $f : X \mapsto \mathbb{R}$ and $g : X \mapsto \mathbb{R}$ be continuous mappings with respect to this metric. Let $h : X \mapsto \mathbb{R}^2$ be the function

$$h(x, \cancel{y}) = (f(x), g(x)).$$

Prove that h is a continuous mapping from (X, d) to $(\mathbb{R}^2, \|\cdot\|_{l^2})$.

Suppose $x_n \to x$ in (X, d). Therefore,

$$|f(x_n) - f(x)| \to 0 \text{ and } |g(x_n) - g(x)| \to 0.$$

Consequently,

$$(f(x_n) - f(x))^2 + (g(x_n) - g(x))^2 \to 0.$$

(b) (25 points) Let $A \in \mathbb{R}^{n \times n}$ be an $n \times n$ real valued matrix and $p \in (1, \infty)$. Prove that the mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by

$$f(\mathbf{x}) = A\mathbf{x}$$

is continuous with respect to the l^p metric on $\mathbb{R}^n$.

Suppose $x_n \to x$ in the ℓ^p metric. Therefore,

$$\|Ax_n - Ax\|_p = \|A(x_n - x)\|_p \leq \|A\|_p \cdot \|x - x_n\|_p.$$

Consequently, $\|Ax_n - Ax\|_p \to 0.$

4. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

(a) (25 points) For $1 \leq p < \infty$ and $k \in \{0, 1, 2, \ldots\}$ let $l^{p,k}(\mathbb{N}, \mathbb{R})$ be the set of sequences $\mathbf{x} = (x_1, x_2, \ldots)$ satisfying

$$\sum_{i=1}^{\infty} \left(1 + i^p + i^{2p} + \ldots i^{kp}\right) |x_i|^p < \infty.$$

This is a normed linear space with metric:

$$\|\mathbf{x} - \mathbf{y}\|_{p,k} = \left(\sum_{i=1}^{\infty} \left(1 + i^p + i^{2p} + \ldots i^{kp}\right) |x_i - y_i|^p\right)^{\frac{1}{p}}.$$

- Prove that $l^{p,k}(\mathbb{N}, \mathbb{R}) \subset l^p(\mathbb{N}, \mathbb{R})$.
- Find a sequence that belongs to $l^{2,1}(\mathbb{N}, \mathbb{R})$ but not $l^{2,2}(\mathbb{N}, \mathbb{R})$.

1. Let $x \in \ell^{p,k}(\mathbb{N}, \mathbb{R})$. Then,

$$\sum_{i=1}^{\infty} |x_i|^p \leq \sum_{i=1}^{\infty} (1 + i^p + \ldots + i^{kp}) |x_i|^p < \infty.$$

2. Let $x \in \ell^2(\mathbb{N}, \mathbb{R})$ be defined by

$$x_i = \frac{1}{i^{5/2}}.$$

Therefore,

$$\|x\|_{2,1}^2 = \sum_{i=1}^{\infty} (1 + i^2) \cdot \frac{1}{i^5} \leq \sum_{i=1}^{\infty} \frac{2i^2}{i^5} = \sum_{i=1}^{\infty} \frac{2}{i^3} < \infty$$

$$\|x\|_{2,2}^2 = \sum_{i=1}^{\infty} (1 + i^2 + i^4) \cdot \frac{1}{i^5} \geq \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

(b) (25 points) Let X be the set of all sequences with only finitely man nonzero terms. (X, d) is a metric space with the metric $d : X \times X \mapsto \mathbb{R}$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{l^\infty}.$$

- Prove that $X \subset l^\infty(\mathbb{N}, \mathbb{R})$.
- Is (X, d) a complete metric space? Prove or provide a counterexample.

- Let $x \in X$. Since x has finitely many nonzero terms it is obvious that $\|x\|_\infty < \infty$.
- No, let $x^{(n)} \in X$ be defined by

$$x_i^{(n)} = \begin{cases} 1/i & \text{if } i \le n \\ 0 & \text{if } i > n \end{cases}$$

Consequently, for $m, n \in \mathbb{N}$ satisfying $m \ge n$ it follows that

$$\|x^{(m)} - x^{(n)}\|_\infty = \frac{1}{n+1}$$

Hence, $x^{(n)}$ is Cauchy. However $x^{(n)} \to (1, 1/2, 1/3, \dots) \notin X$.