Homework 10

Analysis

Due: April 16, 2018

1. (a) Construct a smooth function $f_1 : \mathbb{R} \mapsto \mathbb{R}$ that has the following properties:
    - $\overline{\text{supp}(f_1)} = [-1, 1]$.
    - $\displaystyle\int_{-\infty}^{\infty} f_1(x)\,dx = 1$.
    - $f_1(x) \geq 0$.

   (b) Let $n \in \mathbb{N}$. Show that $f_n : \mathbb{R} \mapsto \mathbb{R}$ defined by $f_n(x) = n f_1(nx)$ satisfies:
    - $\overline{\text{supp}(f_n)} = \left[-\frac{1}{n}, \frac{1}{n}\right]$.
    - $\displaystyle\int_{-\infty}^{\infty} f_n(x)\,dx = 1$.

   (c) Let $g(x)$ be a smooth function. Prove that
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\,dx = g(0).$$

   (d) Let $a, b \in \mathbb{R}$ satisfy $a < b$. Suppose $g \in C^1([a, b])$ satisfies
$$\int_a^b g(x) f(x)\,dx = 0$$
   for all smooth functions $f : \mathbb{R} \mapsto \mathbb{R}$ with compact support contained in $[a, b]$. Prove that $g = 0$.

2. (a) Let X, Y be complete normed linear spaces. Prove that a linear operator $L : X \mapsto Y$ is bounded if and only if $\|L\|_{op} < \infty$.

   (b) Let X, Y be complete normed linear spaces. Prove that if L is a bounded linear operator then for all $x \in X$,
$$\|Lx\|_Y \leq \|L\|_{op} \|x\|_X.$$

3. Let X, Y be complete normed linear spaces. Prove that a linear operator $L : X \mapsto Y$ is continuous at every point in its domain if and only if it is continuous at 0.

4. Let $1 < p < \infty$ and suppose $q \in (1, \infty)$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in L^q([0, 1])$ and define $L : L^p([0, 1]) \mapsto \mathbb{R}$ by
$$L(u) = \int_0^1 u(x) v(x)\,dx.$$
   Prove that L is a bounded linear operator.

5. Let $\delta : C([0, 1]) \mapsto \mathbb{R}$ be the linear operator that evaluates a function at the origin: $\delta(f) = f(0)$.

   (a) If $C([0, 1])$ is equipped with the norm $\|\cdot\|_\infty$ prove that δ is bounded and compute its norm.

   (b) If $C([0, 1])$ is equipped with the norm $\|\cdot\|_{L^1}$ prove that δ is unbounded.

6. Define $K : C([0,1]) \mapsto C([0,1])$ by

$$K(f(x)) = \int_0^1 k(x,y) f(y)\, dy,$$

where $k : [0,1] \times [0,1] \mapsto \mathbb{R}$ is continuous and $k(x,y) \geq 0$. Prove that K is bounded and

$$\|K\|_{op} = \max_{0 \leq x \leq 1} \left\{ \int_0^1 k(x,y)\, dy \right\}.$$